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THE PROPAGATION OF ELECTRIC CURRENTS

IN TELEPHONE AND TELEGRAPH CONDUCTORS

BY

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PREFACE

This book is a reproduction, with some amplifications, of the notes prepared by the Author for two Courses of Postgraduate Lectures given by him before the University of London in the Pender Electrical Laboratory in 1910 and 1911, on the Propagation of Electric Currents in Telephone and Telegraph Conductors and on Electrical Measurements in connection with Telephonic and Telegraphic work. These Lectures had their origin in a request made to the University to provide a course of instruction for Telegraphic and Telephonic Engineers which should enable them to keep abreast of the most recent scientific and technical researches in these branches of Electrical Technology.

These Lectures were attended by a large class composed chiefly of practical Telegraphic and Telephonic Engineers and experts; and at the request of many who attended, and some who did not, the Author has written them out for publication.

As a considerable portion of the subject-matter included has not yet found its way into text-books, although distributed through various technical Journals and Proceedings, it seemed probable that a service would be rendered to Electrical Engineers generally if this material were collected and placed in an easily accessible form. Students of this subject are well aware of the great value of the pioneer work of Mr. Oliver Heaviside and of Prof. Pupin in laying the sound theoretical and practical foundations for improvements of great importance in telephony, and of the classical labours of Lord Kelvin in connection with submarine telegraphy. But the study of the writings of these originators makes a demand for mathematical knowledge which is generally beyond the attainments of the practical telegraphic and telephonic engineer. Prof. A. E. Kennelly has rendered them, however, an immense service in elaborating mathematical methods simple in character and capable of being applied in practical calculations. Much of Prof. Kennelly's instructive expositions are, however, contained in periodicals and journals not very readily obtained by British telegraphists or readers.

The Author has accordingly provided in the first place a simple mathematical introduction which will enable any technical student to acquire easily a working knowledge of the mathematical operations and processes required in conducting the necessary calculations in connection with this subject. In the next place he has endeavoured to simplify as far as possible the theoretical treatment; and thirdly, by illustrative examples, to render it possible for every such student to carry out readily the arithmetic calculations by means of hyperbolic functions in accordance with the methods which have been admirably elucidated by Prof. Kennelly in numerous papers.

The Author desires, in conclusion, to return thanks to those who have assisted or furnished information. Major O'Meara, C.M.G., Engineer-in-Chief of the General Post Office, has most kindly permitted copious extracts and the loan of diagrams from his paper read in 1911 before the Institution of Electrical Engineers, describing the Loaded Anglo-French Telephone Cable laid in 1910. Mr. F. Gill, M.Inst.E.E., Engineer-in-Chief of the National Telephone Company, not only lent apparatus from the investigation laboratory of the National Telephone Company for illustrating the Lectures as given, but has kindly furnished information embodied in many of the tables in this book, and also permitted special measurements to be made in his research laboratory by Mr. B. S. Cohen. The Author desires to record his particular thanks to Prof. A. E. Kennelly, of Harvard University, for permitting a free use to be made of all his valuable papers and writings on this subject and the appropriation of many useful tables such as the Tables of Hyperbolic Functions of Complex Angles in Chapter I. and the Table of Hyperbolic Functions in the Appendix. Papers published by Messrs. Cohen and Shepherd, and read before the Institution of Electrical Engineers, have also been laid under contribution, and to them an acknowledgment is due. Mr. H. Tinsley also

kindly furnished the results of special measurements made with artificial cables, and also granted the use of diagrams of apparatus made by his firm. The Author desires also to include in the list of those who have assisted him, Mr. G. B. Dyke, B.Sc., who aided him efficiently in the Lectures by taking a practical exercise class, and has also made or checked many of the calculations and assisted in reading the proofs of the book. In the hope, therefore, that these republished lectures may be useful to a larger number of telegraphists and telephonists than those to whom they were actually delivered, they are presented in book form, and may serve at least as a stepping stone or introduction to the work of original investigators of a more advanced or difficult character.

J. A. F.

University College,
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THE PROPAGATION OF ELECTRIC CURRENTS IN TELEPHONE AND TELEGRAPH CONDUCTORS

CHAPTER I

MATHEMATICAL INTRODUCTION

1. Introductory Ideas and Definitions. — The object of these lectures is to discuss in as simple a manner as possible the phenomena connected with the propagation of electric currents in telephone and telegraph conductors. This discussion is intended to provide telegraph and telephone engineers with some necessary information to enable them to follow the original writings of leading investigators, and also with the means of solving for themselves practical problems in connection with the subject.

Broadly speaking, the chief scientific problem which presents itself for solution in connection with this matter is that of calculating the current at any time and place in a linear conductor of length very great in comparison with its diameter, when an electromotive force of known type and magnitude is applied at some point in it. Associated with this is the investigation of the effects produced by varying the nature of the conductor and of the terminal apparatus upon the current so transmitted.

The conductors we shall consider may be either bare overhead wires, underground or submarine cables, or telephone wires or cables of different kinds. These conductors, in any case, have four specific qualities which may be reckoned per unit of length, say per mile or per kilometre.

These qualities are—

(i.) The resistance of the conductor per unit of length (R).

(ii.) The inductance of the conductor per unit of length (L).

(iii.) The electrical capacity per unit of length taken with reference to the earth or some other conductor (C).

(iv.) The insulation resistance of the dielectric surrounding the conductor per unit of length, or its reciprocal the insulation conductivity (S).

The above quantities are all of the type called *scalar*, that is they are completely defined as to amount by reference to a unit of the same kind.

It is usual to reckon the resistance in ohms per mile or kilometre, the inductance in henrys or millihenrys per mile or kilometre, the capacity in microfarads per mile or kilometre, and the insulation resistance in megohms per mile or kilometre, or conversely the insulation conductance in the reciprocal of megohms per mile or kilometre, viz., in mhos per mile or kilometre. We have then to consider the current and electromotive force at any point in the conductor. We may specify either their instantaneous values, that is the value they have at any instant, or if they vary cyclically we may specify some function of their instantaneous values throughout the period.

The instantaneous value of the current at any point in the conductor is measured by the ratio of the quantity of electricity dq which flows across the section of the conductor at that point in any time dt to that interval of time, when the interval is taken exceedingly small. If i denotes the current at any instant and dq the quantity of electricity which flows past any section of the conductor in the time dt, then we have

$$i=\frac{dq}{dt}=\dot{q} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The letter q with a dot over it signifies the time rate of change of q. If, however, the current varies in any manner, but so that it passes through a cycle of values in the time T, called the periodic time, then the insertion of a hot wire ammeter in the circuit at that point will give us a reading which is proportional to the square root of the mean of the squares of the instantaneous values of the current taken at small and numerous equidistant intervals of time.

This function of the instantaneous values is called the *root-mean-square value* or the *R.M.S.* value of the current. Mathematically it is expressed by the equation

$$R.M.S.\text{ value of } i=\sqrt{\frac{1}{T}\int_o^T i^2\,dt} \quad . \quad . \quad . \quad (2)$$

As a rule we are not much concerned with the true arithmetic mean value of the instantaneous current throughout a period.

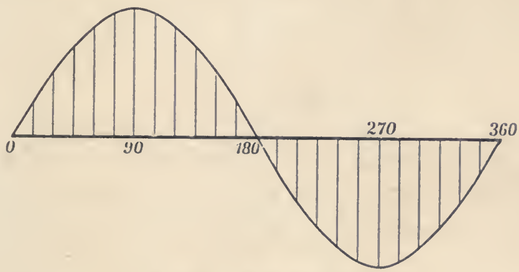

FIG. 1.—A Sine Curve.

When, however, we do have to mention it, it will be denoted by the symbols *T.M.* value of i which is otherwise expressed

$$T.M.\text{ value of } i=\frac{1}{T}\int_o^T i\,dt \quad . \quad . \quad . \quad (3)$$

In a large number of problems the current either varies or can be assumed to vary as the ordinates of a simple curve of sines.

Take any straight line to represent the periodic time and divide it say into 24 parts. At successive points set up lines proportional in length to Sin 0°, Sin 15°, Sin 30°, etc. Join the top of these lines by a smooth curve and we have the curve called a *sine curve* (see Fig. 1). In this way two or more sine curves may be drawn differing in amplitude or *maximum value* and in *phase* or zero point (see Fig. 2).

Taking the point on the left hand at which the ordinate has its zero value we can reckon the abscissa of any point on the curve as equal to an interval of time t on the same scale that the whole period is equal to T. Hence this abscissa reckoned as an angle in circular measure is denoted by $2\pi\frac{t}{T}$ the periodic time being denoted as an angle by 2π. It is usual to write p for $\frac{2\pi}{T}$, and hence the abscissa of any point on the sine curve may be represented by pt in angular measure.

If the ordinate is denoted by i and the maximum ordinate by I we have then the equation to the sine curve in the form

$$i = I \operatorname{Sin} pt \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the origin from which we reckon our time is not the zero point of the curve, but some point more to the left of it, such as

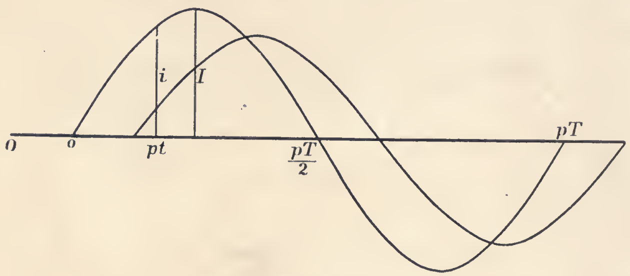

FIG. 2.—Sine Curves differing in phase.

the point O in Fig. 2, then the equation to the two curves in that diagram may be written

$$\begin{aligned} i_1 &= I_1 \operatorname{Sin}(pt - \phi_1) \\ i_2 &= I_2 \operatorname{Sin}(pt - \phi_2) \end{aligned} \quad . \quad . \quad . \quad . \quad (5)$$

The angles ϕ_1 and ϕ_2 are called the phase angles of the zero point and the angle $\phi_1 - \phi_2$ is called the *difference of phase* of the curves.

It is clear, therefore, that to fix the position and form of these curves we require to know two parameters for each, viz., the maximum value I and the phase angle ϕ relative to some point.

We can represent the curve in another manner.

Suppose a line OP of length equal to the maximum value I to revolve round one extremity like the hand of a clock but in a counter-clockwise direction (see Fig. 3). Then if we reckon angles from a fixed line OQ so that $QOM = \phi$ and $QOP = pt$ and hence $MOP = pt - \phi$, it is clear that the projection of OP on the vertical OY, viz., Op, is equal to

$$OP \text{ Sin } (pt - \phi) = I \text{ Sin } (pt - \phi) = i.$$

Accordingly the magnitude of the projection Op which represents the instantaneous value of the current or electro-

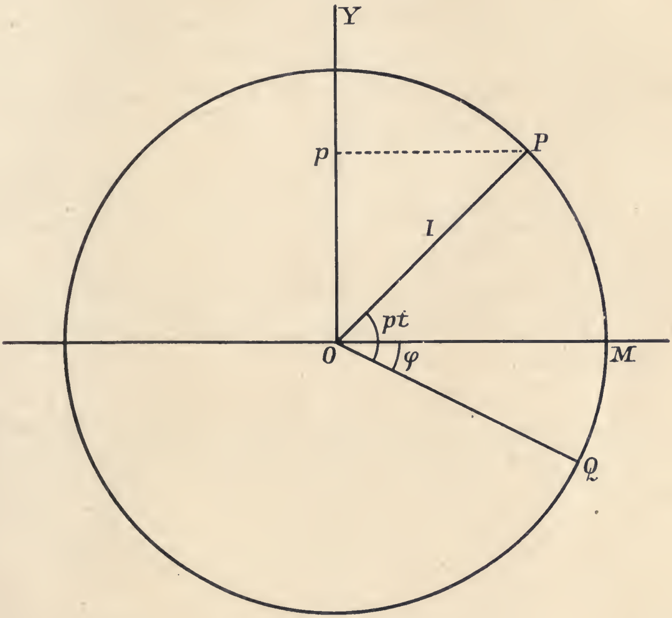

FIG. 3.—Clock Diagram.

motive force is determined by the length of the line OP and its slope at the corresponding instant.

Hence an alternating or simple periodic current which varies from instant to instant proportionately to the ordinates of a sine curve can be represented by a radial line drawn in a certain position on a clock diagram as above described.

It can easily be shown that the mean value of $\text{Sin}^2\,\theta$ taken at equidistant numerous intervals of the angle θ throughout a period or between $\theta = 0°$ and $\theta = 360°$ is equal to $\frac{1}{2}$.

For $$\text{Sin}^2\,\theta = \tfrac{1}{2} - \tfrac{1}{2} \text{ Cos } 2\,\theta.$$

Now the mean value of Sin θ or Cos θ throughout one period or from 0° to 360° is zero; because for every positive value of the ordinate of the curve representing these functions there is an equal negative value. Therefore the mean value of $\frac{1}{2}$ Cos 2 θ throughout a period or from $\theta = 0°$ to $\theta = 360°$ is zero, and therefore the mean value of $\text{Sin}^2\,\theta$ is $\frac{1}{2}$. Therefore the root-mean-square value of the ordinate of a sine curve is $\frac{I}{\sqrt{2}}$ where I is the maximum value. In a clock diagram, therefore, if the revolving radii represent maximum values of the currents or *E.M.F.*, dividing them by $\sqrt{2}$ gives the *R.M.S.* values, assuming that they follow a simple sine law.

We shall see later on that any wave form may be resolved into the sum of a number of sine and cosine curves, and that therefore certain propositions which are true of sine curves are true also of periodic curves of any kind.

For the present, however, we may limit ourselves to the consideration of simple periodic electric currents represented by a simple sine curve.

2. The Representation of Simple Periodic Currents by Complex Quantities.—Having seen that a simple periodic current may be represented by the projection of a revolving radius on a diametral line through the centre of revolution, we have next to consider how such a line can be algebraically specified.

Suppose we draw two lines at right angles through any point, one horizontal and one vertical, we can with the usual conventions as to signs represent by $+ a$ any horizontal line a units in length drawn to the right starting from the origin. Also by $- a$ any horizontal line drawn to the left.

How then shall we represent a line a units in length drawn vertically through the origin upwards or downwards? We can do this by making use of some symbol which shall denote that the horizontal line $+ a$ is turned through a right angle round its left extremity in a counter-clockwise or clockwise direction. This symbol must be such that when prefixed to the symbol a it denotes a line drawn vertically upwards through the origin.

Also it must be such a symbol that when twice repeated it converts $+a$ into $-a$, since turning the horizontal line through two right angles reverses its direction. Let j be this symbol. Then ja is to signify a line of a units in length drawn vertically upwards through the origin or the line a turned through one right angle. Hence jja or j^2a must signify a horizontal line $+a$

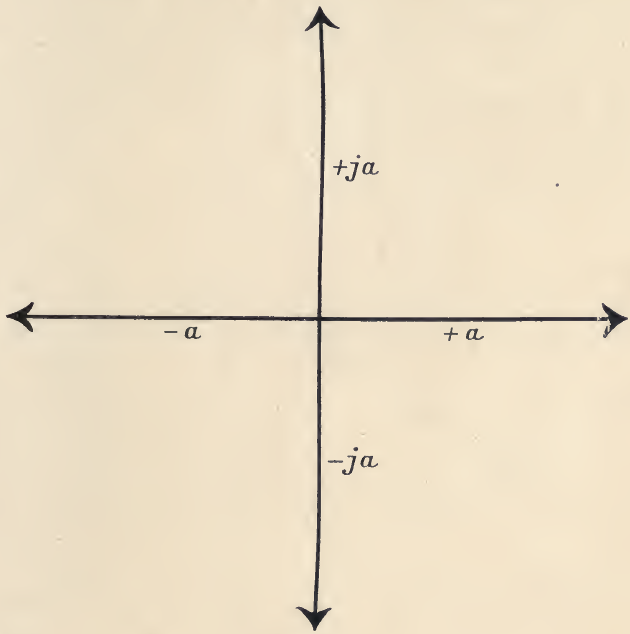

FIG. 4.

turned through two right angles or reversed in direction. Therefore, $j^2a = -a$, and hence $j = \sqrt{-1}$.

The symbol j therefore considered as an operator or sign of an operation is equivalent in meaning to $\sqrt{-1}$.

We have then the following symbols. A line of a units in length drawn horizontally from an origin is denoted by $+a$, a line of the same length drawn vertically upwards is denoted by ja, a line of the same length drawn to the left is $-a$, and an equal line drawn vertically downwards is $-ja$ (see Fig. 4).

If then we give to the sign of addition (+) an extended meaning to make it signify *joint effect*, we can say that the expression $a + jb$ signifies a straight line drawn from any point in such a direction that its horizontal projection is a and its vertical projection is b (see Fig. 5).

For the expression $a + jb$ instructs us to measure a length a starting from the origin in a horizontal direction. Then to measure off a length b in a vertical position starting from the end of a, and the joint effect of these two steps is the same as if we had moved over a straight line of length $\sqrt{a^2 + b^2}$ inclined at an angle θ to the horizontal such that $\tan\theta = \frac{b}{a}$. The quantity $a + jb$ equivalent to $a + \sqrt{-1}\, b$ is called a *complex quantity*,

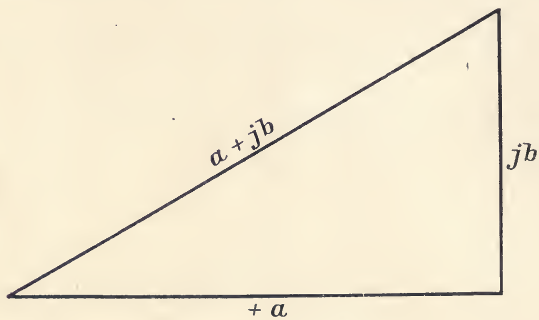

FIG. 5.

and $\sqrt{a^2 + b^2}$ is called its *modulus* or *size*, and $\theta = \tan^{-1}\frac{b}{a}$ its *slope*.

The part a is called the *horizontal step* and b is called the *vertical step*. Hence, $a + jb$ stands for a straight line or anything which has magnitude and direction, such as a force, velocity, or acceleration. In other words, $a + jb$ stands for a *vector* quantity; whilst $\sqrt{a^2 + b^2}$ denotes its size, or mere magnitude apart from direction. We shall in future, following a common custom, denote vectors considered as vectors by letters printed in thick or Clarendon type. Thus **A** signifies a vector or stands for $a + jb$. We shall denote the mere size or modulus by an ordinary Roman capital. Thus A stands for $\sqrt{a^2 + b^2}$. It is more convenient sometimes to denote the mere size or length of a vector A by brackets, *e.g.* (A). The student should note that $a+jb$ signifies not merely a line drawn from one origin, but any line of the same length and with the same slope drawn from any point in the same direction.

We have seen that a simple periodic or alternating electromotive force or current can be represented by a radial straight line the length of which is proportional to the maximum value of or amplitude of the periodic quantity and its slope to the phase with respect to some instant of time. Accordingly such a simple periodic current or E.M.F. can be denoted by a complex quantity such as $a + jb$. The amplitude of the quantity will be measured by $\sqrt{a^2 + b^2}$ and its *R.M.S.* value by $\sqrt{\frac{a^2+b^2}{2}}$. We have then to consider the rules for handling complex quantities in calculations.

3. The Calculus of Complex Quantities.—Let $\mathbf{A} = a + jb$ and $\mathbf{B} = c + jd$ be two complex quantities or vectors; then if $\mathbf{A} = \mathbf{B}$ it signifies that the vectors or lines representing them are equal and parallel. Accordingly, if we draw these lines and set off their horizontal and vertical steps (see Fig. 6), it is clear that the triangles so formed are similar and the side A is equal to the side B. Hence we have also $a = c$ and $b = d$. In other words, *if two complexes are equal we may equate the horizontal and vertical steps respectively.*

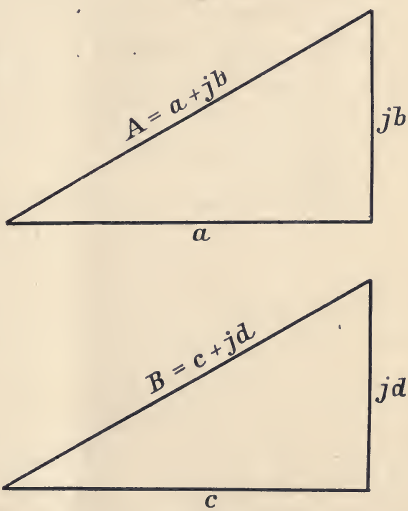

FIG. 6.

In the next place let us consider the result of adding together two complexes. In this process addition is equivalent to joint effect. The complexes represent lines and must be added, therefore, like forces, by the parallelogram law.

If $a+jb$ and $c+jd$ are two complexes representing lines OA, OB drawn from the origin, then their resultant or vector sum is OD, the diagonal of the parallelogram formed on them, It is clear, therefore, from Fig. 7 that OD is a vector whose horizontal step is $a+c$ and vertical step $b+d$. Hence

$$a+jb+c+jd=a+c+j\,(b+d).$$

The second rule is then

To add together two complexes, add the respective horizontal

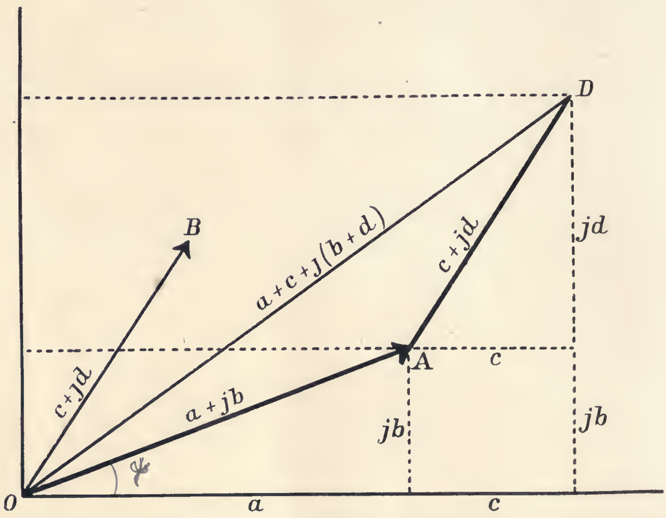

FIG. 7.—Addition of Vectors.

steps for the resultant horizontal step, and the respective vertical steps for the resultant vertical step.

Ex.—Add together $5+j\,6$ and $7+j\,9$. *Ans.* $12+j\,15$.

The same process may be extended to any number of complexes. If a_1+jb_1, a_2+jb_2, etc., are several vectors, then their vector sum is $\Sigma a+j\Sigma b$, where Σa stands for the algebraic sum of all the horizontal steps and Σb of all the vertical steps. It follows that, if the vector sum is zero and if the lines be taken to represent *forces*, these forces are in equilibrium; also that the sides of a polygon taken in order are parallel and proportional to these forces in equilibrium.

Example.—Give expressions in complex form for the sides of a hexagon.

Ans.—Let one side be horizontal and of length a. The next side is represented by $\frac{a}{2}+j\frac{\sqrt{3}}{2}a$, the third by $-\frac{a}{2}+j\frac{\sqrt{3}}{2}a$, the fourth by $-a$, the fifth by $-\frac{a}{2}-j\frac{\sqrt{3}}{2}a$, and the sixth by $\frac{a}{2}-j\frac{\sqrt{3}}{2}a$. The vector sum is zero. Hence forces parallel and proportional to the sides of a hexagon taken in order are in equilibrium.

As a preliminary to additional propositions we must exhibit other expressions for complex quantities. If $a+jb$ is a complex and θ its slope, then obviously $a = \text{A Cos }\theta$ and $b = \text{A Sin }\theta$. Hence we have

$$a+jb=\mathbf{A}=\text{A }(\text{Cos }\theta+j\text{ Sin }\theta).$$

The quantity A is the size of the vector or is $\sqrt{a^2+b^2}$. The quantity $(\text{Cos }\theta+j\text{ Sin }\theta)$ is called a rotating operator or rotator. The effect of it when applied to a vector quantity is to turn the vector through an angle θ without altering its size. Thus $\sqrt{a^2+b^2}$ represents a length or line set off in a horizontal direction; but $\sqrt{a^2+b^2}\,(\text{Cos }\theta+j\text{ Sin }\theta)$ is a line of the same length making an angle θ with the horizontal. Hence any expression of the form $\text{A }(\text{Cos }\theta+j\text{ Sin }\theta)$ represents a line of length A and slope θ.

We can easily prove that the modulus or size of the complex quantity $(a+jb)(\text{Cos }\theta+j\text{ Sin }\theta)$ is the same as the modulus of $a+jb$, viz. $\sqrt{a^2+b^2}$, but the slope of the former vector is greater than that of the latter by an angle θ.

For $(a+jb)\,(\text{Cos }\theta+j\text{ Sin }\theta)=(a\text{ Cos }\theta-b\text{ Sin }\theta)$
$+j\,(b\text{ Cos }\theta+a\text{ Sin }\theta)$.

Now the size of the latter complex is

$$\sqrt{(a\text{ Cos }\theta-b\text{ Sin }\theta)^2+(b\text{ Cos }\theta+a\text{ Sin }\theta)^2}=\sqrt{a^2+b^2}$$

and the slope of this vector is an angle ϕ whose tangent is

$$\frac{b\text{ Cos }\theta+a\text{ Sin }\theta}{a\text{ Cos }\theta-b\text{ Sin }\theta}=\frac{\frac{b}{a}+\text{Tan }\theta}{1-\frac{b}{a}\text{ Tan }\theta}.$$

Hence $\tan\phi = \dfrac{\tan\psi + \tan\theta}{1 - \tan\psi\tan\theta}$ where $\tan\psi = b/a$. Accordingly the slope of $(a + jb)$ (Cos $\theta + j$ Sin θ) is greater than the slope of $a + jb$ by an angle θ, but the sizes are the same.

It is proved in books on trigonometry that

$$\text{Sin } \theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2j}$$

and

$$\text{Cos } \theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2}$$

where ϵ is the base of the Napierian logarithms or the number 2·71828 and j signifies $\sqrt{-1}$.

These are called the exponential values of the Sine and Cosine, and should be committed to memory. If we substitute these values in the expression Cos $\theta + j$ Sin θ we obtain $\epsilon^{j\theta}$. Hence the following are all equivalent expressions for a vector, or complex quantity, viz., $a + jb$, A (Cos $\theta + j$ Sin θ), $A\,\epsilon^{j\theta}$ and A/θ, and they signify a line of length $A = \sqrt{a^2 + b^2}$ and slope $\theta = \tan^{-1}\dfrac{b}{a}$.

The reader should practise himself in converting from one form to the other.

Ex.—Given $3 + j\,4$. Convert to the other forms.

Answer.—The size is $\sqrt{3^2 + 4^2} = 5 = A$ and $\theta = \tan^{-1}\dfrac{4}{3}$ $= 53°\ 7'\ 30''$ nearly. Hence Cos $\theta = 0.6$, and Sin $\theta = 0.8$. Therefore $5\,(0.6 + j\,0.8)$ and $5\,\epsilon^{j\,(53°\ 7'\ 30'')}$ or $5/53°\ 7'\ 30''$ are equivalent to the given expression $3 + j\,4$.

We have next to consider the multiplication of two or more complexes. If $a + jb = A\,\epsilon^{j\theta}$ is one complex and $a_1 + jb_1 = A_1\,\epsilon^{j\theta_1}$ is another, then the products $(a + jb)\ (a_1 + jb_1) = A\,A_1\,\epsilon^{j(\theta + \theta_1)}$. The rule then is, multiply the sizes of the vectors and add the slopes. Thus the product of $a + jb$ and $a_1 + jb_1$ is a vector of which the size is $\sqrt{a^2 + b^2}\ \sqrt{a_1{}^2 + b_1{}^2}$ and the slope is an angle whose tangent is ϕ such that

$$\frac{\dfrac{b}{a} + \dfrac{b_1}{a_1}}{1 - \dfrac{b}{a} \cdot \dfrac{b_1}{a_1}} = \text{Tan } \phi.$$

It follows that the quotient of one complex quantity by another is obtained by the rule, divide the sizes and subtract the angles. For if $A\ \epsilon^{j\theta}$ is one vector and $A_1\ \epsilon^{j\theta_1}$ is the other, then

$$\frac{A\ \epsilon^{j\theta}}{A_1\ \epsilon^{j\theta_1}}=\frac{A}{A_1}\ \epsilon^{j(\theta-\theta_1)}.$$

Again, a complex is reciprocated by reciprocating the size and reversing the angle. For

$$\frac{1}{A\ \epsilon^{j\theta}}=\frac{1}{A}\ \epsilon^{-j\theta}=\frac{1}{A}\ \epsilon^{j(-\theta)}.$$

Also we obtain any power of a complex by the rule, raise the size to that power and multiply the slope by that power. Thus if $A\ \epsilon^{j\theta}$ is a complex then its square is $A^2\ \epsilon^{j2\theta}$ and its square root is $\sqrt{A}\ \epsilon^{j\frac{\theta}{2}}$ and n^{th} power is $A^n\ \epsilon^{jn\theta}$ and n^{th} root is $A^{\frac{1}{n}}\ \epsilon^{j\frac{\theta}{n}}$.

It will be seen, then, that addition and subtraction are most easily carried out when the complexes are in the typical form $a + jb$, but multiplication, division, and raising to powers or extracting roots when the complex is in the form $A\ \epsilon^{j\theta}$. Accordingly it is constantly necessary to convert from one form to the other for calculation.

If we have any function of complex quantities formed of the products, powers, quotients, or roots of complex quantities such as

$$\frac{(a+jb)}{(c+jd)}\,(e+jf)\,(x+jy)^n \quad . \quad . \quad . \quad . \quad (1)$$

it is not necessary to go through the laborious process of reducing it to the canonical form $A + jB$ and to find the size $\sqrt{A^2 + B^2}$. It follows at once from the rules already given that the size of the product of two complexes is the product of their respective sizes, also that the size of any power of a complex is the same power of its size, and hence the size of the quotient of two complexes is the quotient of their sizes. It is quite easy to prove by actual multiplication that the size of the vector $(a + jb)\ (c + jd)$ is $\sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2}$, or is the product of the sizes of the separate vectors.

Also that the size of $\dfrac{a+jb}{c+jd}$ is $\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$. Hence we can write down at once the size of the complex function (1), for it is

$$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}\ \sqrt{e^2+f^2}\ (x^2+y^2)^{\frac{n}{2}} \quad . \quad . \quad . \quad (2)$$

The reader should work the following exercises to familiarise himself with these complex calculations.

Ex. 1.—Draw the two vectors $3 + j\,4$ and $6 + j\,8$ and give their product and quotient of the last by the first in the forms $(a + jb)$ and $\sqrt{a^2 + b^2}\,\underline{/\theta}$.

Ans.—The first is a line of length 5 sloping at an angle $\tan^{-1}\frac{4}{3} = 53^\circ\ 7'\ 30''$, and the second is a line of length 10 at the same angle. Hence they are represented by $5\underline{/53^\circ\ 7'\ 30''}$ and $10\underline{/53^\circ\ 7'\ 30''}$. Their product is a line $50\underline{/106^\circ\ 15'}$, and their quotient is a horizontal line of length 2. Hence their product is $-14 + j\,48$ and quotient $2 + j\,0$.

Ex. 2.—What is the size of the vector $\sqrt{\dfrac{R + jp\,L}{S + jp\,C}}$?

Ans. $\left(\dfrac{R^2 + p^2\,L^2}{S^2 + p^2\,C^2}\right)^{\frac{1}{4}}$.

Ex. 3.—Find the square root of the vector $60 + j\,80$ in the form $A\underline{/\theta}$.

Ans.—$10\underline{/26^\circ\ 33'\ 45''}$.

Ex. 4.—Show how to calculate the value of ϵ the base of the Napierian logarithms.

Ans.—By the exponential theorem we have

$$\epsilon^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \text{etc}$$

Hence if $x = 1$

$$\epsilon = 1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \text{etc.}$$

Hence $\epsilon = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \text{etc.} = 2{\cdot}71828\ \ .\ \ .\ \ .$

The reader should notice that each term of the expansion of ϵ^x is the differential of the next succeeding term. Hence it follows that $\frac{d}{dx}(\epsilon^x) = \epsilon^x$ and $\frac{d^n}{dx^n}\epsilon^x = \epsilon^x$.

If we have any vector or complex quantity represented in the form $A\epsilon^{j\theta}$ or $A\epsilon^{jpt}$ where pt is a phase angle and t denotes time, then the successive differential co-efficients with regard to time are obtained by multiplying the function by jp, $-p^2$, $-jp^3$,

$.+p^4$, etc. Also, since the horizontal and vertical steps of the vector are A Cos pt and A Sin pt, which are simple periodic quantities as t continuously increases, it is more convenient to operate in mathematical work with the function $A\epsilon^{jpt}$ and to take this as the symbolical representation of a simple periodic quantity or sine curve alternating current, understanding this to mean that the periodic variation of the horizontal or vertical steps of $A\epsilon^{jpt}$ represents the current at any instant.

We shall see that it considerably simplifies the mathematics of alternating currents to deal only with the maximum values and avoid the cumbersome trigonometrical expressions involved if we deal with the time variations of the current throughout the period. Hence in our discussions an alternating current or electromotive force will be represented by a complex quantity such as $a+jb$ or $A\epsilon^{jpt}$, and this is to mean that the vector or line represented by these complexes is to represent by its length the maximum value and be supposed to revolve round one extremity so that its projection on a vertical line through the origin represents the actual value of the periodic quantity at that instant on the same scale that the line itself which revolves represents the maximum value or amplitude of the alternating current or *E. M. F.*

4. Hyperbolic Trigonometry.—Since many of the mathematical expressions involved in the theory of the flow of alternating currents through cables can be most conveniently presented, for the purposes of arithmetic calculation, in forms involving hyperbolic trigonometry, it is necessary to explain briefly the nature and properties of these functions. Ordinary trigonometry is called circular trigonometry because the mathematical expressions employed, such as Sines and Cosines, are functions of angles expressed in circular measure or in their equivalent in degrees. These quantities may also be regarded as functions of the area of circular sectors. The shaded area in Fig. 8 represents a segment of a circle. The area of this segment is equal to $\frac{1}{2}r^2\theta$, where θ is the angle *PON* in circular measure and r is the radius *OP*. If we call this area u we have $2\,u/r^2 = \theta$. Now Sin $\theta = PM/OP$ and Cos $\theta = OM/OP$.

Hence if we denote PM by y and OM by x,

$$\text{Sin}\,\frac{2\,u}{r^2}=\frac{y}{r} \text{ and } \text{Cos}\,\frac{2\,u}{r^2}=\frac{x}{r}.$$

Accordingly the Sine and Cosine are here seen to be numerical ratios of the sizes of two lines, and these ratios are functions of a certain kind of the area and radius of a circular sector, the

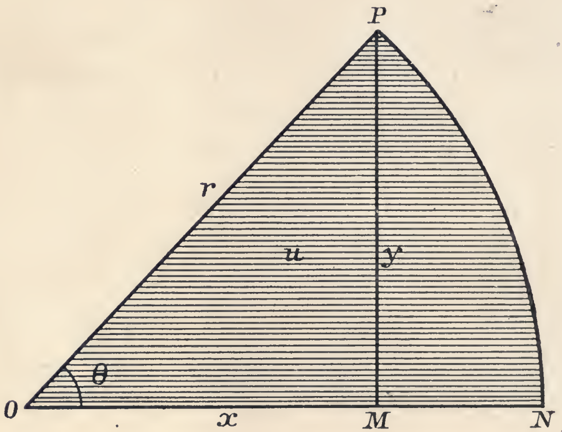

FIG. 8.

said lines being the co-ordinates of the upper point defining the size of the circular sector.

Now the hyperbolic functions with which we shall be concerned are similar functions of the area of the hyperbolic sector of an equilateral hyperbola, and these functions are related to the rectangular hyperbola in the same manner that the ordinary trigonometrical functions are related to the circle.

We shall begin, therefore, by considering the mode of description and the equation of the hyperbola.

The circle is a curve described by a point which moves so that its distance from a fixed point called the centre is constant.

The ellipse is a curve described by a point which moves so that the sum of its distances from two fixed points called the foci is constant.

The hyperbola is a curve described by a point which moves so that the difference of its distances from two fixed points called the foci is constant. Hence it may be described mechanically as follows:—On a sheet of paper take two fixed points F, F' and provide a straight edge rule and a piece of inextensible thread shorter than the rule by a certain amount.

Fix the rule so that one end is pivoted on one of the given points and fasten one end of the thread to the other fixed point

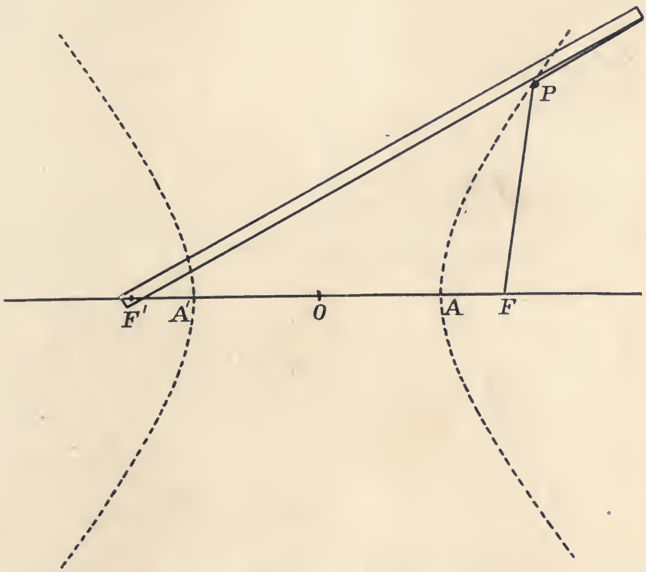

FIG. 9.—Description of an Hyperbola.

and attach the second end of the thread to the free end of the rule. Then press the thread up against the edge of the rule with the point P of a pencil and revolve the rule radially round one fixed point whilst keeping the thread pressed up to its edge by the pencil (see Fig. 9). The point of the pencil will describe one branch of a hyperbola, and the other branch can be described by reversing the attachments of the thread and rule.

The fixed points F and F' (see Fig. 10) are called the foci of the hyperbola, and the points $A\ A'$ where the line $F\,F'$ cuts the branches are called the vertices. The point O bisecting AA' is called the centre. The length OA' is called the semi-major axis and is denoted by a. The distance $OF = OF' = c$ is called the focal distance. The distance $\sqrt{c^2 - a^2} = b$ is called the semi-minor axis. Then $AF = c - a$ and $AF' = c + a$. Hence $AF \cdot AF' = c^2 - a^2 = b^2$. If then P is a point on the hyperbola the difference of the

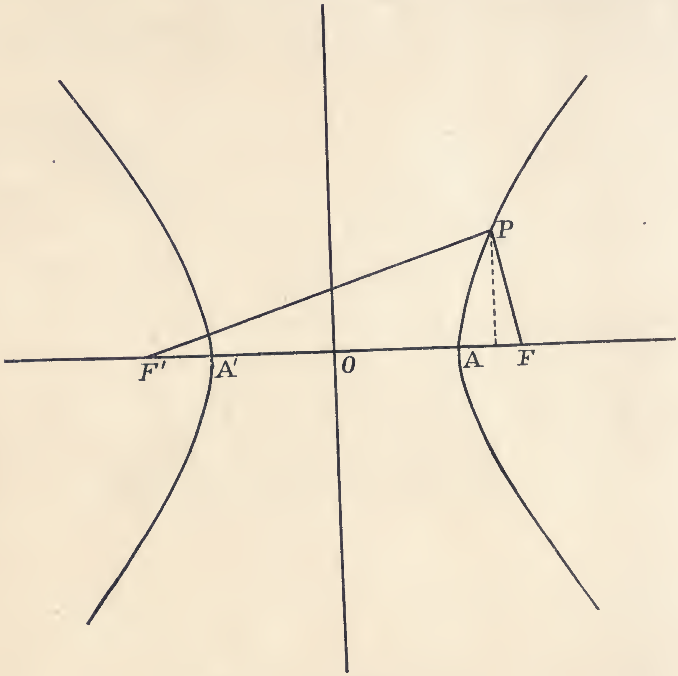

FIG. 10.—An Hyperbola.

distances PF' and PF is constant and is equal to $2a$. Therefore $PF' - PF = 2a$, and if x and y are the co-ordinates of P we have

$$PF = \sqrt{y^2 + (x-c)^2} \text{ and } PF' = \sqrt{y^2 + (x+c)^2}.$$

Therefore $$(PF' + PF)(PF' - PF) = 4cx \quad . \quad . \quad . \quad (3)$$

and $$(PF)^2 + (PF')^2 = 2(y^2 + x^2 + c^2) \quad . \quad . \quad . \quad (4)$$

Accordingly $$PF' + PF = \frac{2cx}{a}, \text{ and } PF' - PF = 2a,$$

or $$PF' = \frac{cx}{a} + a \text{ and } PF = \frac{cx}{a} - a.$$

Substituting these last values of PF and PF' in the equation (4) we have

$$a^2 y^2+(a^2-c^2) x^2 = a^2 (a^2-c^2),$$

or

$$a^2 y^2-b^2 x^2 = -a^2 b^2 \quad . \quad . \quad . \quad . \quad (5)$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This last is the equation to the hyperbola with origin at the centre and rectangular axes through the centre. It is convenient to write it in the form

$$y=\frac{b}{a}\sqrt{x^2-a^2} \quad . \quad (7)$$

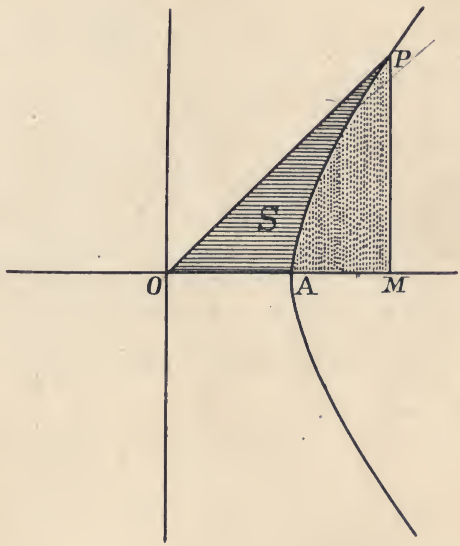

FIG. 11.

We have in the next place to obtain an expression for the area of the hyperbola between the vertex and any ordinate.

The expression for the area of an elementary slice of the hyperbola contained between two ordinates of mean value y separated by a small interval dx is ydx. Hence the area of the hyperbola between the vertex and any abscissa x is obtained when we know the value of the integral $\int_a^x ydx$, or the value of the integral $\frac{b}{a}\int_a^x\sqrt{x^2-a^2}\,dx$.

Let P be any point on the hyperbola (see Fig. 11) and let the dotted area APM be denoted by Δ, then

$$\Delta = \frac{b}{a}\int_a^x \sqrt{x^2-a^2}\,dx \quad . \quad . \quad . \quad . \quad (8)$$

We have then to find the value of the integral $\int\sqrt{x^2-a^2}\,dx$.

Now

$$\int \sqrt{x^2-a^2}\,dx=\int\frac{x^2\,dx}{\sqrt{x^2-a^2}}-a^2\int\frac{dx}{\sqrt{x^2-a^2}} \quad . \quad . \quad (9)$$

Also $$\int \sqrt{x^2-a^2}\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx. \quad . \quad (10)$$

This last is obtained by noting that

$$\frac{d}{dx}\left(x\sqrt{x^2-a^2}\right) = \sqrt{x^2-a^2} + \frac{x^2}{\sqrt{x^2-a^2}}$$

Hence adding (9) and (10) and dividing by 2 we have

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\int \frac{dx}{\sqrt{x^2-a^2}}$$
$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log_e\left\{x + \sqrt{x^2-a^2}\right\}.$$

Therefore we have

$$\frac{b}{a}\int_a^x \sqrt{x^2-a^2}\,dx = \frac{xy}{2} - \frac{ab}{2}\log_e\left\{\frac{x}{a} + \frac{y}{b}\right\} \quad . \quad . \quad (11)$$

If we draw the line OP then the area OAP (shaded) is called the hyperbolic sector and is denoted by S.

It is obvious that the area of the triangle OMP $\left(= \frac{1}{2}xy\right)$ is equal to the sum of S and the dotted area AMP, which we have denoted by Δ, which last is equal to $\frac{b}{a}\int_a^x \sqrt{x^2-a^2}\,dx$. Hence we have

$$S = \frac{1}{2}xy - \Delta = \frac{ab}{2}\log_e\left\{\frac{x}{a} + \frac{y}{b}\right\} \quad . \quad . \quad . \quad (12)$$

If then we consider a rectangular hyperbola or one in which $a = b$ we have

$$\frac{2S}{a^2} = \log_e\left\{\frac{x}{a} + \frac{y}{a}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Finally denoting $\frac{2S}{a^2}$ by u we have

$$\epsilon^u = \frac{x}{a} + \frac{y}{a}.$$

The ratio $\frac{y}{a}$ is called the hyperbolic Sine of u and $\frac{x}{a}$ is called the hyperbolic Cosine of u, and these are written *Sinh* u and *Cosh* u respectively. Therefore

$$\epsilon^u = \text{Cosh}\,u + \text{Sinh}\,u \quad . \quad . \quad . \quad . \quad (14)$$

Now the equation to the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and the equation to the rectangular hyperbola is therefore

$$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$$

or

$$\text{Cosh}^2\, u-\text{Sinh}^2\, u=1 \quad . \quad . \quad . \quad . \quad (15)$$

Dividing this last equation by the equation (14) we have

$$\epsilon^{-u}=\text{Cosh}\, u-\text{Sinh}\, u \quad . \quad . \quad . \quad . \quad (16)$$

and therefore from (14) and (16) we obtain

$$\text{Sinh}\, u=\frac{\epsilon^u-\epsilon^{-u}}{2},\ \text{Cosh}\, u=\frac{\epsilon^u+\epsilon^{-u}}{2} \quad . \quad . \quad (17)$$

We have therefore two definitions of Sinh u and Cosh u which are consistent with each other.

Other hyperbolic functions are defined as follows. The ratio $\frac{\text{Sinh}\, u}{\text{Cosh}\, u}=\frac{y}{x}$ is called the *hyperbolic tangent* and written *Tanh u*. The reciprocal of the hyperbolic Cosine is called the *hyperbolic secant* and written *Sech u*, whilst the reciprocals of the hyperbolic Sine and hyperbolic tangent are called the *hyperbolic cosecant* and *hyperbolic cotangent* and written *Cech u* or *Cosech u* and *Coth u* respectively. Hence we have,

$$\left.\begin{aligned}
\text{Sinh}\, u&=\frac{y}{a}=\frac{\epsilon^u-\epsilon^{-u}}{2}\\
\text{Cosh}\, u&=\frac{x}{a}=\frac{\epsilon^u+\epsilon^{-u}}{2}\\
\text{Tanh}\, u&=\frac{y}{x}=\frac{\epsilon^u-\epsilon^{-u}}{\epsilon^u+\epsilon^{-u}}\\
\text{Cech}\, u&=\frac{a}{y}=\frac{2}{\epsilon^u-\epsilon^{-u}}\\
\text{Sech}\, u&=\frac{a}{x}=\frac{2}{\epsilon^u+\epsilon^{-u}}\\
\text{Coth}\, u&=\frac{x}{y}=\frac{\epsilon^u+\epsilon^{-u}}{\epsilon^u-\epsilon^{-u}}
\end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

These hyperbolic functions are analogous to the corresponding circular functions in ordinary trigonometry, and form the basis of a hyperbolic trigonometry which has many resemblances to it, but is connected with the rectangular hyperbola in place of the circle.

The numerical values of Sinh u, Cosh u, Tanh u, etc., can be calculated for various values of u as follows:—

By the exponential theorem we have

$$\epsilon^{u} = 1 + u + \frac{u^2}{1\,.\,2} + \frac{u^3}{1\,.\,2\,.\,3} + \text{etc.} \quad . \quad . \quad . \ (19)$$

$$\epsilon^{-u} = 1 - u + \frac{u^2}{1\,.\,2\,.\cancel{3}} - \frac{u^3}{1\,.\,2\,.\,3} + \text{etc.} \quad . \quad . \ (20)$$

But $\frac{1}{2}(\epsilon^{u} - \epsilon^{-u}) = $ Sinh u, and hence

$$\text{Sinh}\,u = u + \frac{u^3}{1\,.\,2\,.\,3} + \frac{u^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} + \frac{u^7}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} + \text{etc.}$$

$$= u + \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}} + \frac{u^{\cancel{5}\,7}}{\underline{|7}} + \text{etc.} \quad . \quad . \quad . \quad . \ (21)$$

Similarly since $\frac{1}{2}(\epsilon^{u} + \epsilon^{-u}) = $ Cosh u we have

$$\text{Cosh}\,u = 1 + \frac{u^2}{\underline{|2}} + \frac{u^4}{\underline{|4}} + \frac{u^6}{\underline{|6}} + \text{etc.} \quad . \quad . \quad . \ (22)$$

If therefore we assign any numerical value to u the corresponding values of Sinh u and Cosh u can be calculated with any desired accuracy. Tables of these hyperbolic functions have been calculated and are to be found in many books. A Table of Hyperbolic Sines and Cosines or values of Sinh u and Cosh u from $u = 0$ to $u = 4$ has been calculated by Mr. T. H. Blakesley and is published by Messrs. Taylor and Francis, of Red Lion Court, Fleet Street, London, for the Physical Society of London. A very useful Table of all the Hyperbolic Functions has been constructed by Dr. A. E. Kennelly, based on Ligouski's Tables published in Berlin in

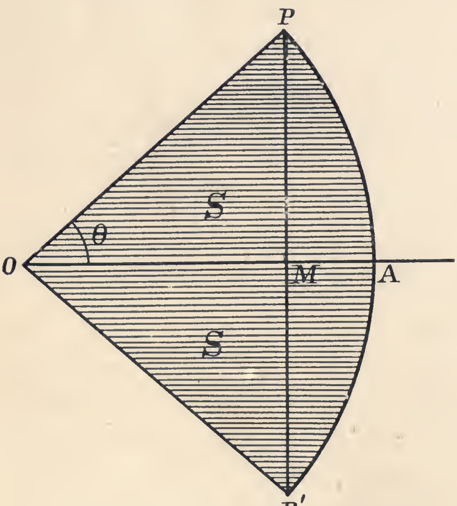

FIG. 12.—Circular Sector.

1890, which by kind permission is reproduced in the Appendix of this book.

Similar Tables are given in Geipel and Kilgour's Electrical Pocket-book, and in a collection of Mathematical Tables arranged by Professor J. B. Dale, published by Messrs. Arnold & Co. Also a small but useful Table of Hyperbolic Functions has been published by Mr. F. Castle, called Five-Figure Logarithms and other Tables (Macmillan & Co., London).

The student should endeavour to obtain a clear idea of the mathematical meaning of these hyperbolic functions and their relation to the ordinary circular trigonometrical functions. This can be done by comparing the diagrams in Fig. 12 and Fig. 13.

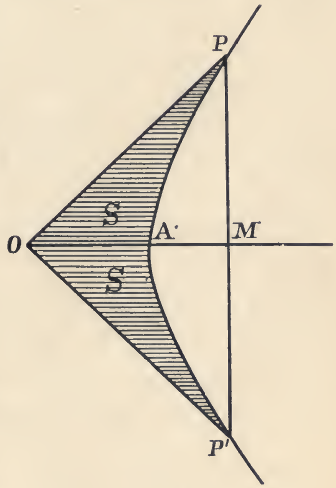

FIG. 13.—Hyperbolic Sector.

In circular trigonometry angles are measured in *radians* or fractions or multiples of a radian. An angle POM is numerically expressed by the ratio of the length of the corresponding circular segment PA to the radius OP of that circle. Hence unit angle or 1 radian is an angle such that the length of the arc is equal to the radius.

The measure of the angle, therefore, is a mere numeric or ratio.

The circular functions Sine, Cosine, etc., are also ratios of lines, viz., the ratio of the vertical projection PM of the radius OP to the radius, or of the horizontal projection OM to the radius OP. These last ratios are considered to be functions of the angle POM. On the other hand the area of the circular segment POA is equal to $\frac{1}{2}$ $(OP)^2$ multiplied by the angle $POA = \theta$ in circular measure. Hence if we call S this area and denote the radius OP by r, then we have

$$\frac{1}{2}r^2\,\theta = S \text{ or } \theta = \frac{2S}{r^2}.$$

If we take the radius r to be unity, then the number which denotes the angle θ is the same as that which measures the area of the circular segment POP'. In other words, if the angle POA is a unit angle in circular measure, then the area of the circular sector POP' is a unit of area in square measure.

The unit angle is equal to 57° 17′ 45″ nearly. Hence if we set off a circular sector with radius 1 cm. and double angle POP' equal to 114° 35′ 30″ the area $APOP'$ will be 1 square centimetre. The circular trigonometrical functions are therefore to be regarded either as functions of the ratio of the arc to the radius or of the area of the segment to the square of the radius.

In the same manner if we draw a rectangular hyperbola and take any point P upon it we can set off a hyperbolic segment $OPAP'$ (shaded area) analogous to the area $OPAP'$ of the circular segment. If the radius OA is taken as unity and if the area of the segment POA' is denoted by S and OA by a, then $\frac{2S}{a^2}$ has been represented by u, and by analogy we may call u the hyperbolic angle.

The reader must carefully distinguish between the hyperbolic measure of an angle and the circular measure of an angle. Thus the circular measure of the angle POA (Fig. 13) may be called θ. Its hyperbolic measure is u.

Now θ is such that $\tan\theta = \frac{y}{x} = \frac{PM}{OM}$ if x and y are respectively PM and OM. But $\frac{y}{a} = \text{Sinh}\, u$ and $\frac{x}{a}$ Cosh u where $a = OA$.

Hence $\frac{y}{x} = \tanh u$, and we have $\tan\theta = \tanh u$.

Thus for instance if the point P is so chosen on the rectangular hyperbola of semi-axis $OA = 1$ that the sector POA has an area of $\frac{1}{2}$ square unit or POP' has an area of 1 unit, then $u = 1$. Now the tables show that for $u = 1$ we have $\tanh u = 0{\cdot}76159$, and also that $\tan 37° \ 17' \ 30'' = 0{\cdot}76159$.

Hence the angle POA in Fig. 13 in ordinary degree measurement is 37° 17′ 30″, and in circular measurement it is 0·651, but in hyperbolic measurement it is unity.

The hyperbolic functions are therefore ratios of lines which are functions of the ratio of the area of a hyperbolic segment to the square of the radius.

5. Formulæ in Hyperbolic Trigonometry.—Just as there are certain relations between the circular functions of ordinary trigonometry, so there are similar formulæ in hyperbolic trigonometry which are of great use.

Fundamental relations in circular trigonometry are

$$\text{Cos}^2\,\theta+\text{Sin}^2\,\theta=1 \qquad (23)$$

$$\text{Sin}\,(a+b)=\text{Sin}\,a\,\text{Cos}\,b+\text{Cos}\,a\,\text{Sin}\,b \qquad (24)$$

$$\text{Cos}\,(a+b)=\text{Cos}\,a\,\text{Cos}\,b-\text{Sin}\,a\,\text{Sin}\,b \qquad (25)$$

From the definitions Sinh $a=\frac{1}{2}(\epsilon^a-\epsilon^{-a})$ and Cosh $a=\frac{1}{2}(\epsilon^a+\epsilon^{-a})$ and similar definitions for Sinh b and Cosh b it is easy to prove by substitution that

$$\text{Cosh}^2\,\theta-\text{Sinh}^2\,\theta=1 \qquad (26)$$

$$\text{Cosh}^2\,\theta+\text{Sinh}^2\,\theta=\text{Cosh}\,2\theta \qquad (27)$$

also that

$$\text{Sinh}\,(a\pm b)=\text{Sinh}\,a\,\text{Cosh}\,b\pm\text{Cosh}\,a\,\text{Sinh}\,b \qquad (28)$$

$$\text{Cosh}\,(a\pm b)=\text{Cosh}\,a\,\text{Cosh}\,b\pm\text{Sinh}\,a\,\text{Sinh}\,b \qquad (29)$$

and hence that

$$\text{Tanh}\,(a\pm b)=\frac{\text{Tanh}\,a\pm\text{Tanh}\,b}{1\pm\text{Tanh}\,a\,\text{Tanh}\,b} \qquad (30)$$

These formulæ are easily verified by substituting for Sinh a, $\frac{1}{2}(\epsilon^a-\epsilon^{-a})$, and for Cosh a, $\frac{1}{2}(\epsilon^a+\epsilon^{-a})$, and the same for Sinh b and Cosh b. It will be seen that the formulæ are identical in form with the corresponding ones in circular trigonometry, but that in some cases algebraic signs are different.

With the aid therefore of a table of hyperbolic Sines and Cosines there is no difficulty in calculating out the results.

It will be well for the reader to plot curves representing the variation of the hyperbolic functions as the hyperbolic angle increases and compare these with the corresponding curves for the circular functions. The curves in Fig. 14 represent the variation of Sinh u, Cosh u, and Tanh u as the angle u increases. These curves therefore are non-periodic and do not repeat themselves like the curves representing Sin θ, Cos θ, Tan θ.

In using hyperbolic trigonometry in connection with the solution of problems on the propagation of electric currents in conductors we shall find that we meet with such expressions as

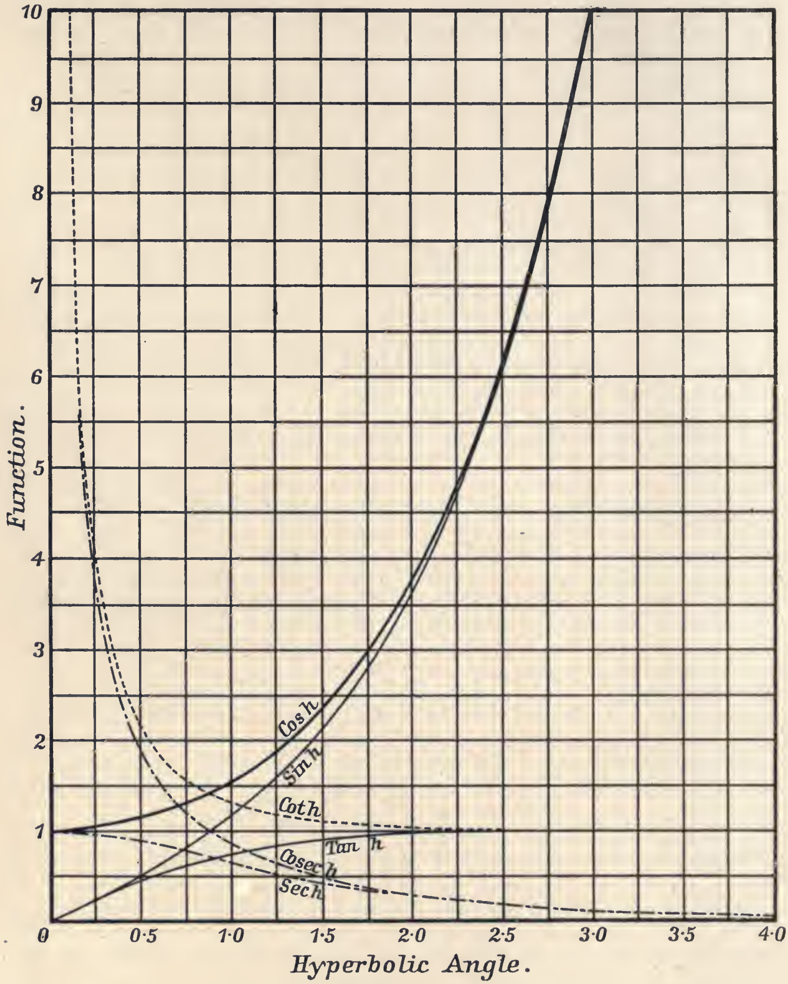

FIG. 14.—Curves representing the variation of the Hyperbolic Functions.

Sinh $(a+jb)$, Cosh $(a+jb)$, etc., where a and b are numerical quantities and j as usual signifies $\sqrt{-1}$. We have then to consider the meaning of such an expression as Cosh ja or Sinh ja.

If we remember that Sin $a = \frac{\epsilon^{ja} - \epsilon^{-ja}}{2j}$ and Cos $a = \frac{\epsilon^{ja} + \epsilon^{-ja}}{2}$ and also that Sinh $u = \frac{\epsilon^{u} - \epsilon^{-u}}{2}$ and Cosh $u = \frac{\epsilon^{u} + \epsilon^{-u}}{2}$ it will be clear that Cosh $ja = \frac{\epsilon^{ja} + \epsilon^{-ja}}{2}$ and therefore that Cos a is identical with Cosh ja. In other words the Cosine of a circular angle is identical with the hyperbolic Cosine of a hyperbolic angle ja. This last expression ja is called an *imaginary angle.*

Hence the Cosine of a real angle is equivalent to the hyperbolic Cosine of an imaginary angle. Again from the exponential values of Sin a and Sinh a it is evident that j Sin a = Sinh ja. In a similar manner the following formulæ can be proved :—

$$\begin{aligned} &\text{Cos } ja = \text{Cosh } a. && \text{Cos } a = \text{Cosh } ja. \\ &\text{Sin } ja = j \text{ Sinh } a. && j \text{ Sin } a = \text{Sinh } ja \\ &\text{Tan } ja = j \text{ Tanh } a. && j \text{ Tan } a = \text{Tanh } ja. \end{aligned} \qquad (31)$$

If then we meet with such an expression as Sinh $(a + jb)$ we can expand it by the ordinary rule and eliminate the hyperbolic functions of the imaginary angles by the aid of the above expressions. Thus

$$\text{Sinh } (a+jb) = \text{Sinh } a \text{ Cosh } jb + \text{Cosh } a \text{ Sinh } jb \qquad (32)$$

or

$$\text{Sinh } (a+jb) = \text{Sinh } a \text{ Cos } b + j \text{ Cosh } a \text{ Sin } b \qquad (33)$$

In the same way we find

$$\text{Cosh } (a+jb) = \text{Cosh } a \text{ Cos } b + j \text{ Sinh } a \text{ Sin } b. \qquad (34)$$

It is evident then that these equivalents for Sinh $(a + jb)$ and Cosh $(a + jb)$ are vector or complex quantities of the form $A + jB$ because the quantities such as Cosh a Cos b and Sinh a Sin b which form the A and B terms are numerical quantities.

Hence the hyperbolic functions of complex angles such as $a + jb$ are vectors, such as Cosh a Cos b $+ j$ Sinh a Sin b.

The quantities $a + jb$ when so used may be called complex hyperbolic angles composed of a real angle and an imaginary angle.

If we divide Sinh $(a + jb)$ by Cosh $(a + jb)$ we have Tanh $a + jb$, and hence

$$\text{Tanh } (a+jb) = \frac{\text{Tanh } a + j \text{ Tanh } b}{1 + j \text{ Tanh } a \text{ Tanh } b} \qquad (35)$$

If we denote the size of the vector Sinh $(a + jb)$ by putting brackets round it thus (Sinh $a + jb$) we have

$$(\text{Sinh } a+jb)=\sqrt{\text{Sinh}^2 a \text{ Cos}^2 b+\text{Cosh}^2 a \text{ Sin}^2 b},$$

but $\quad \text{Cos}^2 b=1-\text{Sin}^2 b$ and $\text{Cosh}^2 a=1+\text{Sinh}^2 a.$

Hence

$$(\text{Sinh } a+jb)=\sqrt{\text{Sinh}^2 a+\text{Sin}^2 b} \quad . \quad . \quad . \quad (36)$$

also

$$(\text{Cosh } a+jb)=\sqrt{\text{Cosh}^2 a-\text{Sin}^2 b} \quad . \quad . \quad . \quad (37)$$

Again the slope of Sinh $(a + jb)$ is an angle ϕ such that

$$\text{Tan } \phi=\text{Coth } a \text{ Tan } b,$$

and of Cosh $a + jb$ is

$$\text{Tan } \phi=\text{Tanh } a \text{ Tan } b.$$

Accordingly if any line or vector $a + jb$ is given drawn on a diagram we can draw other lines or vectors on the same diagram to represent the quantities Sinh $(a + jb)$, Cosh $(a + jb)$, Tanh $(a + jb)$, Sech $(a + jb)$, Cech $(a + jb)$, and Coth $(a + jb)$.

It will be frequently necessary to consider how such functions vary as a or b have different magnitudes, that is to say, as the size and slope of $a + jb$ vary.

For example, find and draw the hyperbolic functions of $1 + 1{\cdot}5j$.

We have

$$\text{Sinh } (1+j\ 1{\cdot}5) = \text{Sinh } 1 \text{ Cos } 1{\cdot}5+j \text{ Cosh } 1 \text{ Sin } 1{\cdot}5.$$

These numbers 1·5 and 1 are therefore angles in circular and hyperbolic measure respectively. Since $\pi = 3{\cdot}1415 = 180^\circ$ the angle in degrees corresponding to 1·5 in circular measure is $180 \times \frac{1{\cdot}5}{3{\cdot}1415} = 89^\circ\ 7'\ 44''$.

Hence $\quad \text{Sin } 1{\cdot}5={\cdot}99988$ and $\text{Cos } 1{\cdot}5={\cdot}01525$

also $\quad \text{Sinh } 1=1{\cdot}17520$ and $\text{Cosh } 1=1{\cdot}54308$

Therefore

$$\text{Sinh } (1+j\ 1{\cdot}5)=1{\cdot}1752 \times {\cdot}01525+j\,(1{\cdot}5431 \times {\cdot}99988)$$

or

$$\text{Sinh } (1+j\ 1{\cdot}5)={\cdot}018+j\ 1{\cdot}543.$$

Hence the size $\quad (\text{Sinh } (1+j\ 1{\cdot}5))=1{\cdot}54$ nearly

and the slope is $\quad \text{Tan}^{-1} \dfrac{1{\cdot}543}{{\cdot}018}=89^\circ$ nearly.

Therefore $\quad \text{Sinh } (1+j\ 1{\cdot}5)=1{\cdot}54 \underline{/89^\circ}.$

In the same manner we can, from the formula

$$\text{Cos}\,(a+jb) = \text{Cosh}\,a\,\text{Cos}\,b+j\,\text{Sinh}\,a\,\text{Sin}\,b,$$

find that $\text{Cos}\,(1+j\,1{\cdot}5) = {\cdot}0235+j\,1{\cdot}175$

Hence the size is 1·38 nearly and the slope 90° or

$$\text{Cosh}\,(1+j\,1{\cdot}5) = 1{\cdot}38\ \underline{/90^\circ}.$$

Therefore $\text{Tanh}\,(1+j\,1{\cdot}5 = 1{\cdot}11\ \overline{/1^\circ}.$

Also $\text{Sech}\,(1+j\,1{\cdot}5) = {\cdot}072\ \overline{/90^\circ}.$

and $\text{Cech}\,(1+j\,1{\cdot}5 = {\cdot}065\overline{/89^\circ}$, and lastly

$$\text{Coth}\,(1+j\,1{\cdot}5) = {\cdot}09\ \underline{/1^\circ}.$$

We can therefore plot out these vectors as in Fig. 15, where the firm lines represent the hyperbolic functions of $0{\cdot}5 + j\,0{\cdot}8$, which are more widely separated than those of $1 + j\,1{\cdot}5$. In this last case the Sinh and Cosh fall so nearly on each other that they cannot be shown as separated lines.

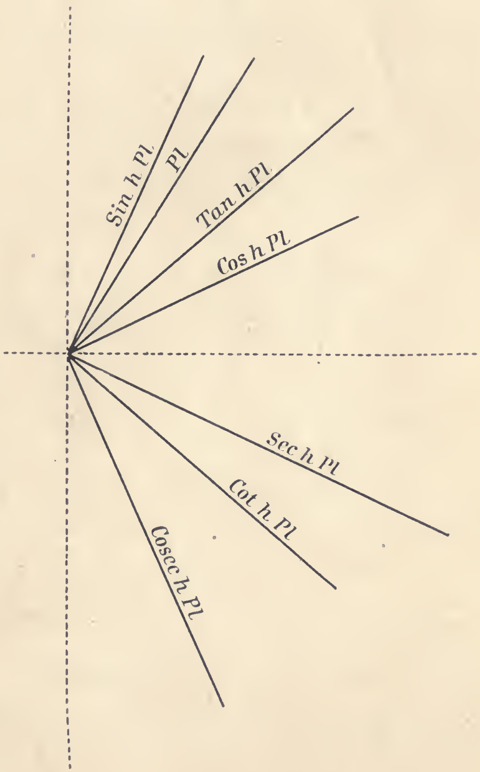

Fig. 15.—Vectors representing Hyperbolic Functions of $Pl = 0{\cdot}5 + j\,0{\cdot}8$.

Again, we may take any given function such as Sinh $(a + jb)$ and give various ratios to $\frac{b}{a}$; that is, we may suppose the vector $a + jb$ to be turned round its end so that whilst retaining the same size it has various slopes, and we may examine the corresponding variation in the hyperbolic functions.

The ordinary logarithms of the hyperbolic functions, that is

$\log_{10}$ (Sinh u), $\log_{10}$ (Cosh u), and $\log_{10}$ (Tanh u), were calculated by Dr. C. Gudermann and published in 1833 at Berlin in a book entitled "Theorie der Potenzial Cyklisch-hyperbolischen Functionen." Unfortunately he only gives these logarithms for values of u between 2 and 12. A copy of the book is in the Graves Library of University College, London. These tables, however, facilitate the calculation of the hyperbolic functions of complex angles, because they enable us to calculate pretty easily the values of Sinh a Cosh b and of Cosh a Sinh b, etc., and hence of Sinh $(a+jb)$, Cosh $(a+jb)$, etc., for values of a and b between 2 and 12.

We can also obtain a graphical construction for the vectors representing these hyperbolic functions of complex angles in the following way.

In the case of an ellipse of eccentricity e and semi-axes a and b, the distance from the centre to either focus being denoted by f, we have the well-known relations

$$\frac{b^2}{a^2}=1-e^2 \text{ or } b^2=a^2(1-e^2) \text{ and } ae=f.$$

Hence by substitution we can put the equation to the ellipse with origin at the centre, viz.: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the form

$$e^2x^2+\frac{e^2y^2}{1-e^2}=f^2 \qquad . \quad . \quad . \quad . \quad (38)$$

If we take f to be unity and select such a hyperbolic angle a that Cosh $a=\frac{1}{e}$, then Sinh $a=\frac{\sqrt{1-e^2}}{e}$, and the equation to the ellipse with origin at centre then takes the form

$$\frac{x^2}{\text{Cosh}^2\,a}+\frac{y^2}{\text{Sinh}^2\,a}=1 \qquad . \quad . \quad . \quad . \quad (39)$$

Again with regard to the hyperbola of eccentricity e_1, and semi-axes a_1 and b_1 we have $-\frac{b_1^2}{a_1^2}=1-e_1^2$,

or $$b_1^2=a_1^2(e_1^2-1) \text{ and } f=a_1e_1.$$

If then the focal distance $f=1$, and if we take such a circular angle β that Cos $\beta=\frac{1}{e_1}$, we can put the central equation of the hyperbola, viz., $\frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=1$, in the form

$$e_1^2 x^2 - \frac{e_1^2 y^2}{e_1^2 - 1} = 1 \quad . \quad . \quad . \quad . \quad (40)$$

or

$$\frac{x^2}{\text{Cos}^2 \beta} - \frac{y^2}{\text{Sin}^2 \beta} = 1 \quad . \quad . \quad . \quad . \quad (41)$$

If then we have an ellipse of eccentricity $e = \frac{1}{a} = \frac{1}{\text{Cosh } \alpha}$ and a confocal hyperbola of eccentricity $e_1 = \frac{1}{a_1} = \frac{1}{\text{Cos } \beta}$ it is clear that they intersect at some point P and that the co-ordinates of

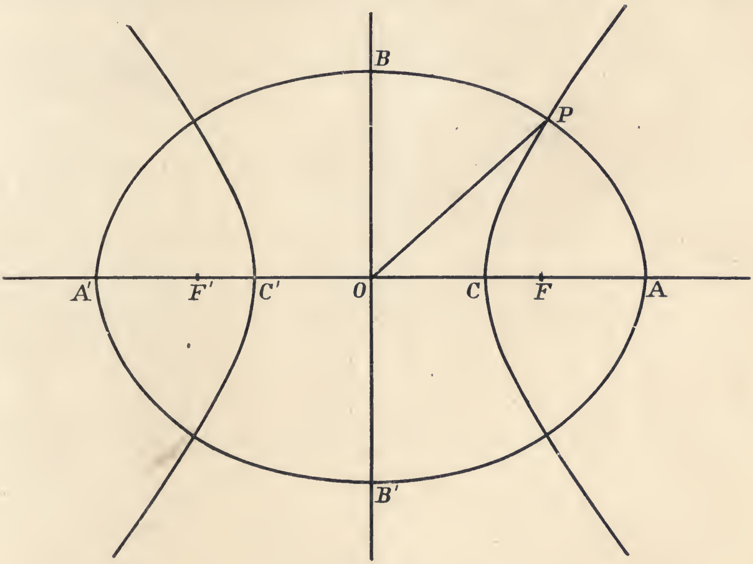

FIG. 16.

this point x and y are obtained by solving as simultaneous equations,

$$\frac{x^2}{\text{Cosh}^2 \alpha} + \frac{y^2}{\text{Sinh}^2 \alpha} = 1 \quad . \quad . \quad . \quad . \quad (42)$$

$$\frac{x^2}{\text{Cos}^2 \beta} - \frac{y^2}{\text{Sin}^2 \beta} = 1 \quad . \quad . \quad . \quad . \quad (43)$$

It is obvious by inspection, having regard to the fact that $\text{Cos}^2 \beta + \text{Sin}^2 \beta = 1$ and $\text{Cosh}^2 \alpha - \text{Sinh}^2 \alpha = 1$, that the solutions of (42) and (43) are,

$$x = \text{Cosh } \alpha \text{ Cos } \beta \quad . \quad . \quad . \quad . \quad (44)$$

$$y = \text{Sinh } \alpha \text{ Sin } \beta \quad . \quad . \quad . \quad . \quad (45)$$

because these satisfy the equations (42) and (43).

The radius vector OP of the point of intersection of the ellipse and hyperbola is expressed as a complex quantity by $x+jy=\text{Cosh}\,\alpha\,\text{Cos}\,\beta+j\,\text{Sinh}\,\alpha\,\text{Sin}\,\beta=\text{Cosh}\,(\alpha+j\beta)$. Accordingly we can set off a line to represent Cosh $(\alpha+j\beta)$ given $\alpha+j\beta$ as follows: Take a horizontal line and any point O in it (see Fig. 16). Set off distances OF OF' on either side of O of unit

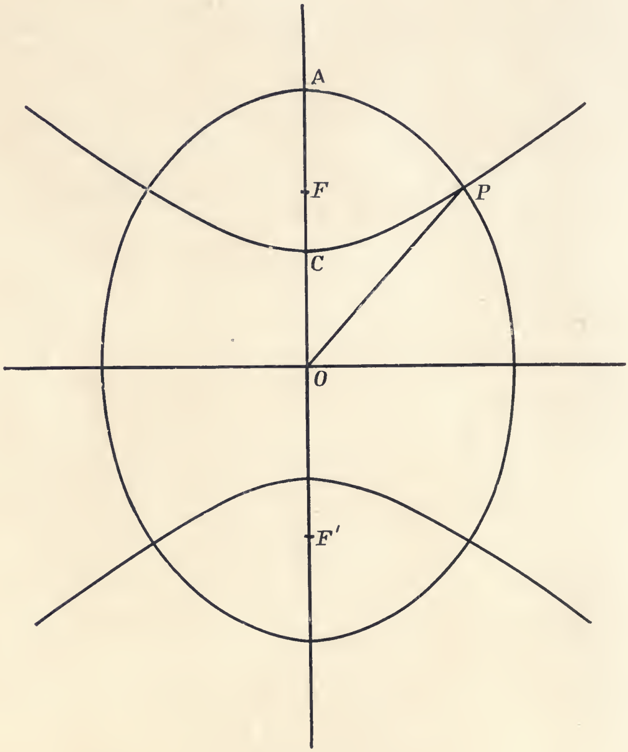

FIG. 17.

length. Set off distances OA, OC representing to the same scale the values of Cosh α and Cos β as given in the Tables. Draw a line OB at right angles to OA and take a point B in it such that $BF=OA$. Then describe an ellipse in the foci F and F' and semi-axes OA, OB. This can be done by making a loop of thread embracing the points F and F', and of length equal to $F'F+FB+BF'$ and moving a pencil point round so as to keep the thread tight. Then describe an hyperbola with the same foci and semi-major axis $OC = \text{Cos}\,\beta$. The line OP represents to scale $\text{Cosh}\,(\alpha + j\,\beta)$ because it is $x + j\,y$, and these have been proved above to be equal to one another. It is well known that confocal ellipses and hyperbolas intersect each other at right angles.

A very similar construction enables us to draw a vector representing $\text{Sinh}\,(\alpha + j\,\beta)$, having given $\alpha + j\,\beta$.

Draw vertical and horizontal lines intersecting at O (see Fig. 17). Set off distances OF', OF equal to unity on the vertical line on either side of O. Set off a distance OA equal to $\text{Cosh}\,\alpha$ to the same scale and a distance OC equal to $\text{Sin}\,\beta$, and with foci F' and F describe an ellipse with semi-major axis OA and an hyperbola with semi-major axis OC. These will intersect at P. Then OP represents $\text{Sinh}\,(\alpha + j\,\beta)$. Let the co-ordinates of P be x and y. Then the equation to the ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ and if e is the eccentricity $\frac{b^2}{a^2} = 1 - e^2$. Also $ae = 1$, and the equation to the ellipse is therefore

$$\frac{e^2\,x^2}{1-e^2} + e^2\,y^2 = 1,$$

but if $a = \frac{1}{e} = \text{Cosh}\,\alpha$, then $\frac{1-e^2}{e^2} = \text{Sinh}^2\,\alpha$ and the equation takes the form

$$\frac{x^2}{\text{Sinh}^2\,\alpha} + \frac{y^2}{\text{Cosh}^2\,\alpha} = 1 \quad . \quad . \quad . \quad . \quad (46)$$

In the same way we can prove that the equation to the confocal hyperbola is

$$\frac{y^2}{a_1{}^2} - \frac{x^2}{b_1{}^2} = 1$$

or

$$e_1{}^2\,y^2 - \frac{e_1{}^2\,x^2}{e_1{}^2 - 1} = 1$$

or

$$\frac{y^2}{\text{Sin}^2\,\beta} - \frac{x^2}{\text{Cos}^2\,\beta} = 1 \quad . \quad . \quad . \quad . \quad (47)$$

The solution of the equations (46) and (47) as simultaneous equations gives us the co-ordinates of the point P of intersection.

It is obvious that the solution is

$$\left.\begin{aligned} x &= \text{Sinh}\,\alpha\ \text{Cos}\,\beta \\ y &= \text{Cosh}\,\alpha\ \text{Sin}\,\beta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (48)$$

Hence

$OP = x + jy = \text{Sinh } a \text{ Cos } \beta + j \text{ Cosh } a \text{ Sin } \beta = \text{Sinh } (a + j\beta)$.

Accordingly OP represents Sinh $(a + j\beta)$ on the same scale that OA = Cosh a and OC represents Sin β.

It is clear that since an ellipse of given foci is defined by its semi-major axis and the same for the confocal hyperbola we might describe a number of confocal ellipses and hyperbolas of different eccentricities and affix to each a numerical value a and β where a is such a quantity that Cosh a numerically measures the semi-major axis of the ellipse and β such a quantity that Cos β represents the semi-major axis of the hyperbola, the focal distance OF for all being unity. Then we can obtain the value of Cosh $(a + j\beta)$ by looking out the ellipse marked a and the hyperbola marked β and joining the point of intersection with the centre, that vector would then represent Cosh $(a + j\beta)$. Such a series of confocal ellipses and hyperbolas has been delineated by Messrs. Houston and Kennelly in a paper entitled "Resonance in Alternating Current Lines," published in the *Transactions of the American Institute of Electrical Engineers*, Vol. XII., April, 1905, p. 208. Dr. Kennelly has also calculated the values of Sinh $(a + j\beta)$, Cosh $(a + j\beta)$, Tanh $(a + j\beta)$, Cech $(a + j\beta)$, Sech $(a + j\beta)$, and Coth $(a + j\beta)$ for fifteen values of $\sqrt{a^2 + \beta^2}$ from 0 to 1·5 and for values of $\frac{\beta}{a}$ equal to 1, 2, 3, 4, 10, and set them out in Tables[1] which by his very kind permission are reproduced here.

Thus, for instance, the Table I. shows us that the hyperbolic sine of a vector $1 \underline{/45^\circ}$ of which the size therefore is unity and ratio β/a is also unity or slope 45° is a vector $1{\cdot}0055 \underline{/54^\circ\ 32'}$, and from Table II. we find that the hyperbolic Cosine of the same vector is a vector $1{\cdot}0803 \underline{/27^\circ\ 29'}$.

These Tables will be found of great use in subsequent calculations.

If then we are given any vector within limits in the form $a + jb$, we can convert it into the form $\sqrt{a^2 + b^2}\ \underline{/\text{Tan}^{-1} b/a}$ and

[1] See Dr. A. E. Kennelly. "The Distribution of Pressure and Current over Alternating Current Circuits," *The Harvard Engineering Journal*, 1905—1906.

TABLE I.

Values of Sinh $(a+jb)$ or Hyperbolic Sines of Vectors of various sizes and slopes given in the form $A/\underline{\theta}$.
Ratio b/a 1 to 10. Size $\sqrt{a^2+b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}=1$.		$=2$.		$=3$.		$=4$.		$=10$.	
$\sqrt{a^2+b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	0·0000	45° 00′	0·0000	63° 26′	0·0000	71° 34′	0·0000	75° 58′	0·000	84° 17′
0·1	0·1000	45° 05′	0·0999	63° 30′	0·0998	71° 34′	0·0999	76° 01′	0·0998	84° 17′
0·2	0·2000	45° 23′	0·1992	63° 44′	0·1988	71° 48′	0·1991	76° 09′	0·1986	84° 22′
0·3	0·2999	45° 51′	0·2973	64° 08′	0·2965	72° 65′	0·2960	76° 22′	0·2955	84° 28′
0·4	0·4000	46° 31′	0·3937	64° 40′	0·3916	72° 52′	0·3907	76° 41′	0·3897	84° 36′
0·5	0·5003	47° 23′	0·4877	65° 22′	0·4835	73° 01′	0·4818	77° 06′	0·4800	84° 46′
0·6	0·6006	48° 27′	0·5789	66° 13′	0·5718	73° 40′	0·5688	77° 38′	0·5655	84° 59′
0·7	0·7010	49° 40′	0·6668	67° 15′	0·6555	74° 27′	0·6508	78° 14′	0·6454	85° 15′
0·8	0·8016	51° 06′	0·7510	68° 27′	0·7341	75° 22′	0·7272	78° 57′	0·7191	85° 33′
0·9	0·9033	52° 44′	0·8310	69° 50′	0·8070	76° 25′	0·7973	79° 47′	0·7865	85° 55′
1·0	1·0055	54° 32′	0·9066	71° 23′	0·8739	77° 37′	0·8605	80° 45′	0·8448	86° 19′
1·1	1·1089	56° 31′	0·9775	73° 08′	0·9355	79° 03′	0·9219	81° 50′	0·8956	86° 47′
1·2	1·2138	58° 41′	1·0435	74° 46′	0·9877	80° 31′	0·9647	83° 03′	0·9377	87° 19′
1·3	1·3205	61° 02′	1·1047	77° 15′	1·0342	82° 13′	1·0049	84° 25′	0·9706	87° 55′
1·4	1·4297	63° 34′	1·1610	79° 38′	1·0730	84° 07′	1·0565	85° 57′	0·9941	88° 35′
1·5	1·5418	66° 15′	1·2125	82° 14′	1·1047	86° 14′	1·0606	87° 42′	1·0081	89° 20′

NOTE.—1 = tan 45°, 2 = tan 63° 30′, 3 = tan 71° 35′, 4 = tan 76°, and 10 = tan 84° 20′.
Hence Sinh $1{\cdot}5/\underline{76°} = 1{\cdot}0606/\underline{87° 42'}$.

TABLE II.

Values of Cosh $(a+jb)$ or Hyperbolic Cosines of Vectors of various sizes and slopes given in the form $A/\underline{\theta}$.
Ratio b/a 1 to 10. Size $\sqrt{a^2+b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}=1$.		$=2$.		$=3$.		$=4$.		$=10$.	
$\sqrt{a^2+b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′
0·1	1·0000	0° 17′	0·9970	0° 14′	0·9960	0° 11′	0·9956	0° 08′	0·9951	0° 04′
0·2	1·0001	1° 09′	0·9881	0° 56′	0·9841	0° 41′	0·9824	0° 33′	0·9805	0° 14′
0·3	1·0007	2° 35′	0·9736	2° 06′	0·9645	1° 35′	0·9607	1° 15′	0·9562	0° 31′
0·4	1·0021	4° 35′	0·9538	3° 47′	0·9375	2° 52′	0·9308	2° 16′	0·9227	0° 57′
0·5	1·0051	7° 08′	0·9295	6° 01′	0·9038	4° 36′	0·8930	3° 38′	0·8801	1° 33′
0·6	1·0107	10° 16′	0·9010	8° 52′	0·8637	6° 51′	0·8479	5° 26′	0·8291	2° 19′
0·7	1·0198	13° 53′	0·8706	12° 22′	0·8184	9° 40′	0·7966	7° 44′	0·7702	3° 20′
0·8	1·0336	18° 00′	0·8385	16° 37′	0·7693	13° 13′	0·7399	10° 39′	0·7040	4° 39′
0·9	1·0533	22° 34′	0·8071	21° 41′	0·7177	17° 38′	0·6791	14° 22′	0·6313	6° 22′
1·0	1·0803	27° 29′	0·7782	27° 36′	0·6656	23° 07′	0·6160	19° 09′	0·5533	8° 41′
1·1	1·1157	32° 41′	0·7542	34° 26′	0·6137	30° 02′	0·5531	25° 19′	0·4712	11° 56′
1·2	1·1608	38° 05′	0·7378	42° 06′	0·5716	38° 08′	0·4935	33° 19′	0·3865	16° 44′
1·3	1·2162	43° 35′	0·7316	50° 28′	0·5370	47° 59′	0·4422	43° 39′	0·3026	24° 21′
1·4	1·2832	49° 05′	0·7376	59° 18′	0·5165	59° 15′	0·4052	56° 34′	0·2250	37° 40′
1·5	1·3616	54° 33′	0·7571	68° 18′	0·5138	71° 23′	0·3894	71° 34′	0·1687	62° 16′

TABLE III.

Values of Tanh $(a+jb)$ or Hyperbolic Tangents of Vectors of various sizes and slopes given in the form $A/\underline{\theta}$.
Ratio b/a 1 to 10. Size $\sqrt{a^2+b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}=1$.		$=2$.		$=3$.		$=4$.		$=10$.	
$\sqrt{a^2+b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	0·0000	45° 00′	0·0000	63° 26′	0·0000	71° 34′	0·0000	75° 58′	0·0000	84° 18′
0·1	0·1000	44° 48′	0·1002	63° 16′	0·1002	71° 23′	0·1004	75° 53′	0·1003	84° 13′
0·2	0·1999	44° 14′	0·2016	62° 48′	0·2020	71° 07′	0·2027	75° 36′	0·2026	84° 08′
0·3	0·2997	43° 16′	0·3054	62° 02′	0·3074	70° 30′	0·3081	75° 07′	0·3091	83° 57′
0·4	0·3991	41° 56′	0·4128	60° 53′	0·4178	70° 00′	0·4197	74° 25′	0·4223	83° 39′
0·5	0·4977	40° 15′	0·5248	59° 21′	0·5350	68° 35′	0·5395	73° 28′	0·5454	83° 13′
0·6	0·5942	38° 11′	0·6425	57° 21′	0·6620	66° 49′	0·6708	72° 12′	0·6820	82° 40′
0·7	0·6874	35° 47′	0·7659	54° 53′	0·8010	64° 47′	0·8170	70° 30′	0·8380	81° 55′
0·8	0·7756	33° 06′	0·8956	51° 50′	0·9542	62° 09′	0·9828	68° 18′	1·0213	80° 54′
0·9	0·8576	30° 10′	1·0295	48° 09′	1·1243	58° 47′	1·1740	65° 25′	1·2460	79° 33′
1·0	0·9306	27° 03′	1·1650	43° 47′	1·3130	54° 30′	1·3967	61° 36′	1·5268	77° 38′
1·1	0·9939	23° 50′	1·2960	38° 42′	1·5242	40° 01′	1·6667	56° 31′	1·9006	74° 51′
1·2	1·0456	20° 36′	1·4141	32° 40′	1·7280	42° 23′	1·9548	49° 44′	2·4258	70° 35′
1·3	1·0857	17° 27′	1·5100	26° 37′	1·9260	34° 14′	2·2722	40° 46′	3·2070	63° 34′
1·4	1·1141	14° 29′	1·5740	20° 20′	2·0777	24° 52′	2·6070	29° 23′	4·4180	50° 55′
1·5	1·1323	11° 42′	1·6015	13° 56′	2·1500	14° 51′	2·7235	16° 08′	5·9750	27° 04′

TABLE IV.

Values of Cosech $(a+jb)$ or Hyperbolic Cosecants of Vectors of various sizes and slopes given in the form $A/\underline{\theta}$.
Ratio b/a 1 to 10. Size $\sqrt{a^2+b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}=1$.		$=2$.		$=3$.		$=4$.		$=10$.	
$\sqrt{a^2+b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
		—		—		—		—		—
0·0		45° 00′		63° 26′		71° 34′		75° 58′		84° 17′
0·1	10·0025	45° 05′	10·009	63° 30′	10·017	71° 34′	10·010	76° 01′	10·021	84° 17′
0·2	5·0011	45° 23′	5·020	63° 44′	5·0301	71° 48′	5·0228	76° 09′	5·035	84° 22′
0·3	3·3340	45° 51′	3·363	64° 08′	3·3732	72° 05′	3·3780	76° 22′	3·384	84° 28′
0·4	2·5002	46° 31′	2·540	64° 40′	2·5540	72° 52′	2·560	76° 41′	2·566	84° 36′
0·5	1·9989	47° 23′	2·0502	65° 22′	2·0680	73° 01′	2·0754	77° 06′	2·083	84° 46′
0·6	1·6651	48° 27′	1·7273	66° 13′	1·7488	73° 40′	1·7580	77° 38′	1·786	84° 59′
0·7	1·4265	49° 40′	1·4997	67° 15′	1·5255	74° 27′	1·5365	78° 14′	1·549	85° 15′
0·8	1·2475	51° 06′	1·3316	68° 27′	1·3622	75° 22′	1·3752	78° 57′	1·391	85° 33′
0·9	1·1070	52° 44′	1·2034	69° 50′	1·2391	76° 25′	1·2543	79° 47′	1·272	85° 55′
1·0	0·9945	54° 32′	1·1030	71° 23′	1·1442	77° 37′	1·1621	80° 45′	1·184	86° 19′
1·1	0·9018	56° 31′	1·0230	73° 08′	1·0689	79° 03′	1·0847	81° 50′	1·117	86° 47′
1·2	0·8238	58° 41′	0·9583	74° 46′	1·0125	80° 31′	1·0366	83° 03′	1·067	87° 19′
1·3	0·7573	61° 02′	0·9052	77° 15′	0·9670	82° 13′	0·9951	84° 25′	1·030	87° 55′
1·4	0·6995	63° 34′	0·8613	79° 38′	0·9319	84° 07′	0·9465	85° 57′	1·006	88° 35′
1·5	0·6486	66° 15′	0·8247	82° 14′	0·9051	86° 14′	0·9434	87° 42′	0·992	89° 20′

N.B.—All the angles in this table are negative.

TABLE V.

Values of Sech $(a+jb)$ or Hyperbolic Secants of Vectors of various sizes and slopes given in the form $A/\underline{\theta}$

Ratio b/a 1 to 10. Size $\sqrt{a^2+b^2}$ 0 to 1·5.

	Ratio $\frac{b}{a}=1$.		$=2$.		$=3$.		$=4$.		$=10$.	
$\sqrt{a^2+b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′	1·0000	0° 00′
0·1	1·0000	0° 17′	1·0030	0° 14′	1·0040	0° 11′	1·0044	0° 08′	1·0049	0° 04′
0·2	0·9999	1° 09′	1·0120	0° 56′	1·0162	0° 41′	1·0179	0° 33′	1·0199	0° 14′
0·3	0·9993	2° 35′	1·0272	2° 06′	1·0369	1° 35′	1·0409	1° 15′	1·0458	0° 31′
0·4	0·9979	4° 35′	1·0485	3° 47′	1·0667	2° 52′	1·0744	2° 16′	1·0838	0° 57′
0·5	0·9949	7° 08′	1·0759	6° 01′	1·1065	4° 36′	1·1199	3° 38′	1·1362	1° 33′
0·6	0·9894	10° 16′	1·1099	8° 52′	1·1579	6° 51′	1·1793	5° 26′	1·2061	2° 19′
0·7	0·9806	13° 53′	1·1487	12° 22′	1·2219	9° 40′	1·2553	7° 44′	1·2984	3° 20′
0·8	0·9675	18° 00′	1·1926	16° 37′	1·2998	13° 13′	1·3516	10° 39′	1·4205	4° 39′
0·9	0·9494	22° 34′	1·2390	21° 41′	1·3930	17° 38′	1·4726	14° 22′	1·5840	6° 22′
1·0	0·9256	27° 29′	1·2850	27° 36′	1·5025	23° 07′	1·6234	19° 09′	1·8073	8° 41′
1·1	0·8963	32° 41′	1·3259	34° 26′	1·6296	30° 02′	1·8081	25° 19′	2·1222	11° 56′
1·2	0·8614	38° 05′	1·3553	42° 06′	1·7496	38° 08′	2·0262	33° 19′	2·5873	16° 44′
1·3	0·8222	43° 35′	1·3669	50° 28′	1·8623	47° 59′	2·2613	43° 39′	3·3047	24° 21′
1·4	0·7793	49° 05′	1·3557	59° 18′	1·9360	59° 15′	2·4677	56° 34′	4·4703	37° 40′
1·5	0·7344	54° 13′	1·3208	68° 18′	1·9463	71° 23′	2·5682	71° 34′	5·9277	62° 16′

N.B.—All the angles in this table are negative.

TABLE VI.

Values of Coth $(a+jb)$ or Hyperbolic Cotangents of Vectors of various sizes and slopes given in the form $A/\underline{\theta}$.
Ratio b/a 1 to 10. Size $\sqrt{a^2+b_2}$ 0 to 1·5.

	Ratio $\frac{b}{a}=1$.		$=2$.		$=3$.		$=4$.		$=10$.	
$\sqrt{a^2+b^2}$.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.	Size A.	Angle θ.
0·0		45° 00′		63° 26′		71° 34′		75° 58′		84° 18′
0·1	10·0025	44° 48′	9·980	63° 16′	9·980	71° 23′	9·960	75° 53′	9·970	84° 13′
0·2	5·0018	44° 14′	4·960	62° 48′	4·950	71° 07′	4·933	75° 36′	4·936	84° 08′
0·3	3·3365	43° 16′	3·274	62° 02′	3·253	70° 30′	3·246	75° 07′	3·235	83° 57′
0·4	2·5050	41° 56′	2·422	60° 53′	2·393	70° 00′	2·383	74° 25′	2·370	83° 39′
0·5	2·0092	40° 15′	1·905	59° 21′	1·869	68° 35′	1·854	73° 28′	1·834	83° 13′
0·6	1·6830	38° 11′	1·556	57° 21′	1·511	66° 49′	1·491	72° 12′	1·466	82° 40′
0·7	1·4547	35° 47′	1·306	54° 53′	1·2484	64° 47′	1·224	70° 30′	1·193	81° 55′
0·8	1·2894	33° 06′	1·117	51° 50′	1·0480	62° 09′	1·018	68° 18′	0·9791	80° 54′
0·9	1·1660	30° 10′	0·9714	48° 09′	0·8894	58° 47′	0·8518	65° 25′	0·8026	79° 33′
1·0	1·0746	27° 03′	0·8584	43° 47′	0·7616	54° 30′	0·7160	61° 36′	0·6550	77° 38′
1·1	1·0061	23° 50′	0·7716	38° 42′	0·6560	49° 01′	0·6000	56° 31′	0·5261	74° 51′
1·2	0·9564	20° 36′	0·7071	32° 40′	0·5787	42° 23′	0·5116	49° 44′	0·4122	70° 35′
1·3	0·9211	17° 27′	0·6623	26° 37′	0·5192	34° 14′	0·4401	40° 46′	0·3118	63° 34′
1·4	0·8996	14° 29′	0·6353	20° 20′	0·4815	24° 52′	0·3836	29° 23′	0·2263	50° 55′
1·5	0·8831	11° 42′	0·6244	13° 56′	0·4651	14° 51′	0·3672	16° 08′	0·1674	27° 04′

N.B.—All of the angles in the table are negative.

look out in these Tables the hyperbolic functions and thus determine Sinh $(a + jb)$, Cosh $(a + jb)$, etc., in the form of vectors expressed as $A \underline{/\theta}$, etc.

We sometimes require an expression for an inverse hyperbolic function such as $\text{Cosh}^{-1}(a+jb)$. Since this quantity is a vector it must have such a value that

$$\text{Cosh}^{-1}(a+jb)=x+jy,$$

or

$$\text{Cosh}(x+jy)=a+jb.$$

Hence $$a+jb=\text{Cosh}\,x\,\text{Cos}\,y+j\,\text{Sinh}\,x\,\text{Sin}\,y.$$

Equating vertical and horizontal steps we have

$$a=\text{Cosh}\,x\,\text{Cos}\,y$$

$$b=\text{Sinh}\,x\,\text{Sin}\,y.$$

But $$\text{Sin}^2 y+\text{Cos}^2 y=1 \text{ and } \text{Cosh}^2 x-\text{Sinh}^2 x=1.$$

Therefore by substitution we find

$$\frac{a^2}{\text{Cosh}^2 x}+\frac{b^2}{\text{Sinh}^2 x}=1$$

or

$$\frac{a^2}{\text{Cosh}^2 x}+\frac{b^2}{\text{Cosh}^2 x-1}=1.$$

Multiplying up we arrive at a biquadratic equation

$$\text{Cosh}^4 x-\text{Cosh}^2 x+a^2=a^2\,\text{Cosh}^2 x+b^2\,\text{Cosh}^2 x$$

which can be written in the form,

$$\left\{\text{Cosh}^2 x-\frac{a^2+b^2+1}{2}\right\}^2=\left(\frac{a^2+b^2+1}{2}\right)^2-a^2.$$

Hence

$$\text{Cosh}^2 x=\frac{a^2+b^2+1}{2}\pm\sqrt{\left(\frac{a^2+b^2+1}{2}\right)^2-a^2}$$

$$=\frac{2\,(a^2+b^2+1)\pm 2\sqrt{(a^2+b^2+1)^2-4\,a^2}}{4}.$$

This last expression can be put in the form

$$\text{Cosh}^2 x=\frac{(1+a)^2+b^2+(1-a)^2+b^2\pm 2\sqrt{((1+a)^2+b^2)\,((1-a)^2+b^2)}}{4}$$

which is an exact square. Therefore

$$\text{Cosh}\,x=\frac{\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}}{2} \quad . \quad . \quad (49)$$

In the same manner we can show that

$$\text{Cos}\,y=\frac{\sqrt{(1+a)^2+b^2}.\pm\sqrt{(1-a)^2+b^2}}{2}.$$

Accordingly

$$\operatorname{Cosh}^{-1}(a+jb)=\operatorname{Cosh}^{-1}\frac{\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}}{2}$$
$$+j\operatorname{Cos}^{-1}\frac{\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}}{2} \quad . \quad . \quad (50)$$

And by a similar process we can prove that

$$\operatorname{Sinh}^{-1}(a+jb)=\operatorname{Cosh}^{-1}\frac{\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}}{2}$$
$$+j\operatorname{Sin}^{-1}\frac{\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}}{2} \quad . \quad . \quad (51)$$

These formulæ have important applications.

CHAPTER II

THE PROPAGATION OF ELECTROMAGNETIC WAVES ALONG WIRES

1. Wave Motion.—As the subject-matter of these lectures is an exposition of the effects connected with the propagation of electromagnetic waves along wires, it may be well to commence by some explanation of the nature of wave motion generally. Let us consider a material medium like the air composed of discrete particles or atoms which we shall, for the sake of simplicity, assume to be initially at rest. The medium has two fundamental mechanical qualities. It possesses *Inertia* in virtue of which any particle of it when set in motion tends to persist in that motion unless compelled to change its motion by impressed force. This is equivalent to stating that when any mass M of the medium is moving without rotation with a velocity V it possesses kinetic or motional energy measured by $\frac{1}{2}MV^2$. Also the medium possesses some kind of *Elasticity*—that is it resists change of form or shape or motion. In the case of a fluid like air the elasticity is resistance to change of volume of a given mass. It resists compression or expansion. In consequence of these two qualities inertia and elasticity the medium permits the propagation through it of *wave motion*. This means that any change in the medium made suddenly at one place is not instantly reproduced or repeated at all points, but makes its appearance successively at different points. Thus, if in an unlimited mass of air we cause a sudden increase in pressure of the air at one spot by heating it, say by an electric spark, the surrounding air does not immediately relieve this pressure by moving outwards everywhere at once, because in virtue of the inertia of the air the force due to the initial compression cannot immediately create outward motion in the surrounding shell of air. When, however, the immediately surrounding layer of air has been set in motion outwards it relieves the pressure at the origin, and the original state of compression is now transferred to a shell of air embracing the original region of compression. This process again repeats itself, and the state of compression is handed on to a still larger spherical shell or layer, and thus the original state of compression is propagated outwards in the form of a spherical shell of compression which changes its locus progressively by continually increasing its size.

Whilst the general body of the air remains undisturbed this thin spherical region or shell in which the air is compressed

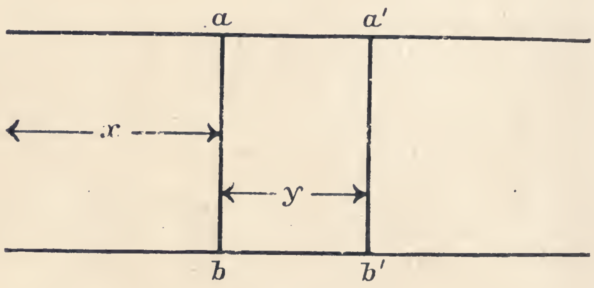

FIG. 1.

continually becomes greater in radius and forms what is called a wave of compression in the air.

The characteristic of wave motion is therefore that the particular kind of disturbance (in this case compression) is repeated successively and not simultaneously at all points of the medium. If we take two points in the medium separated by a certain distance x and note the time interval t between the appearance of the disturbance at these places, then x/t is called the *wave velocity* (W). This wave velocity depends upon the specific qualities of the medium, viz., its density or inertia per unit of volume and its elasticity.

To fix our ideas let us consider waves of longitudinal displacement such as sound waves travelling up a tube of unit cross-section filled with air. The particles of air lying on any section of the tube will then move to and fro together. Let the density or mass of air per unit volume be denoted by ρ, and its elasticity or the ratio of compressing force or pressure to the corresponding compression in volume be denoted by e. Then if dp is the increment of pressure causing a reduction of volume dv in a volume of air v, we have $e = -\frac{vdp}{dv}$. Consider a layer of air particles lying on a section $a\,b$ of the air in the tube (see Fig. 1). Let x denote their distance from a fixed section at zero time, and let $x + y$ be their distance after a time t as the wave of longitudinal displacement moves over them. Then y is the displacement in the time t of the particles which form this section ab.

Suppose then that we fix our attention upon a slice of the air bounded by two planes at distances x and $x + \delta x$ from the origin. As the wave passes over this slice the sections of it are moved so that the particles which were initially at x are moved to $x + y$, and the particles which were initially at $x + dx$ are moved to

$$x+y+\delta\,(x+y)=x+\delta x+y+\delta y.$$

Hence the thickness of the slice which was originally δx becomes $\delta x + \delta y$. Its increase in volume is therefore δy, and the ratio of increase of volume to original volume is $\frac{\delta y}{\delta x}$, or ultimately, when δx is very small, it becomes $\frac{dy}{dx}$.

If the changes in pressure of the slice of air are made very slowly, then the product of pressure p and volume v of a unit of mass is constant, which may be expressed by the formula $pv =$ a constant. If, however, the compression is very suddenly applied so that the heat due to the compression remains in the slice and augments its pressure or elasticity, then the relation of p and v is given by the equation $pv^a = a\ constant$ where $a = 1{\cdot}41$ nearly, and is the ratio of the specific heat at constant pressure to the specific heat at constant volume. This is the case in an air wave. Hence we have by differentiation of $pv^a =$ constant, $dpv^a + av^{a-1}\,pdv = 0$ or $dp = -\,ap\,\frac{dv}{v}$ or $-\,\frac{vdp}{dv} = e = -\,ap.$

The force moving the slice of air of thickness δx is the difference of pressure on its two surfaces, viz., the value of

$\frac{d}{dx}(dp)\delta x$. But $dp = -\alpha p\frac{dv}{v}$, and we have shown that for the air motion here considered we have $\frac{dv}{v} = \frac{dy}{dx}$.

Hence the moving force on the air section is

$$-\alpha p\frac{d^2 y}{dx^2}\delta x = e\frac{d^2 y}{dx^2}\delta x.$$

The displacement of the slice being y, it follows that its acceleration is $\frac{d^2y}{dt^2}$, and since its mass is $\rho\delta x$, the equation of motion is

$$\rho\delta x\frac{d^2 y}{dt^2} = e\delta x\frac{d^2 y}{dx^2}$$

or

$$\frac{d^2 y}{dt^2} = \frac{e}{\rho}\frac{d^2 y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The above is a type of *differential equation* which presents itself very frequently in Physics. It is not difficult to show that it is satisfied by any value of y which is made up of the sum of any single valued functions of $x - \sqrt{\frac{e}{\rho}}\,t$ and $x + \sqrt{\frac{e}{\rho}}\,t$.

So that

$$y = F\left(x - \sqrt{\frac{e}{\rho}}\,t\right) + F\left(x + \sqrt{\frac{e}{\rho}}\,t\right) \quad . \quad . \quad (2)$$

Any function such as $F\left(x - \sqrt{\frac{e}{\rho}}\,t\right)$ represents a wave of wave-form $y = F(x)$ travelling forward with a velocity $W = \sqrt{\frac{e}{\rho}}$. For the function $F\left(x - \sqrt{\frac{e}{\rho}}\,t\right)$ has the same value if for x we substitute $x - x'$, and for t, $t - t'$, provided that $x'/t' = \sqrt{\frac{e}{\rho}}$. The reader should carefully consider the physical meaning of this statement.

Any function of x such as $y = F(x)$ represents a stationary curve whose ordinate y at any point is some function of its abscissa x. It therefore represents a wave-form. If the curve moves bodily forward without change of shape with a speed W, then the ordinate having a value y at a time t corresponding to an abscissa x has the same height as the ordinate y at a distance

$x + dx$ corresponding to a time $t + dt$, provided $\frac{dx}{dt}$ is the velocity W of the wave.

In other words, the characteristic of a wave motion is that the same state is repeated at a distance dx ahead at a time dt ahead provided $\frac{dx}{dt}$ is the velocity of disturbance. Hence any mathematical function such as $F(x - Wt)$ for which this is true represents a wave advancing with a velocity W.

Accordingly for a medium of density ρ and elasticity e the wave velocity W is

$$W=\sqrt{\frac{e}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2. The Electromagnetic Medium.—It is now almost universally agreed that the phenomena of electricity and magnetism render it necessary to postulate an electromagnetic space-filling medium or æther, and it has been shown that what we call light and radiant heat as well as electric radiation are waves of a particular kind in this medium. Moreover, a large body of proof exists tending to show that the elements of material substance described as atoms are built up of constituents called negative electrons or corpuscles and of positive electrons; and that these negative electrons collectively constitute so-called negative electricity. The reader desirous of placing himself *au courant* with what is known and believed on these matters may be referred to the following excellent works for a full exposition of them:—"Electricity and Matter," by Sir J. J. Thomson (Archibald Constable & Co., London); "A Treatise on Electrical Theory," by G. W. de Tunzelmann (Charles Griffin & Co.); "The Electron Theory," by E. E. Fournier d'Albe (Longmans, Green & Co.); "Electromagnetic Theory," by Oliver Heaviside (The Electrician Publishing Company, London). The advanced reader will do well to consult "Æther and Matter," by Sir Joseph Larmor (Cambridge University Press), and "The Theory of Electrons," by H. A. Lorentz (David Nutt & Co., London).

The sum and substance of the scientific creed taught by these writers is that the basis for all physical phenomena as well as the source of all gravitative Matter is to be found in the properties of the Universal Æther, and that not only Matter but also Electricity has an atomic structure, and that the atoms of electricity, or, to speak more correctly, of negative electricity, are the electrons which are the constituents of the chemical atom.

The hypothesis has been advanced that the electron itself is a strain centre or focus of certain lines of strain in the æther of a particular kind. Hence the movement of the electron is merely a displacement of the strain form from one place to another in a stagnant æther. Experimentally it is established that an electron is a small charge of negative electricity assumed to be distributed over a small sphere having a diameter about one hundred thousandth of that of a chemical atom. It is therefore a centre on which converge lines of electric force. The phenomena of electricity and magnetism prove that in the neighbourhood of electrified bodies there is a distribution along curved or straight lines of *electric strain*, which strain is a physical state of the material dielectric or the interpenetrating æther. This state is also called *electric displacement* or *polarisation*. Similarly near magnetic poles and conductors carrying electric currents there is a distribution of *magnetic flux* or *induction*.

The magnetic flux and electric strain are particular states of the æther or matter occupying the field, which are vector quantities having direction as well as magnitude at each point in the field. Thus the electron is a centre of converging lines of electric strain, and a wire conveying an electric current is embraced by endless lines of magnetic flux. The important question then arises whether these "lines of force" are merely mathematical abstractions like lines of latitude and longitude or whether we are to regard them as having objective existence. Arguments of a weighty character have been advanced by Sir J. J. Thomson for the view that these lines of magnetic and electric force are not merely directions in the field, but, so to speak, structures which compose it.[1] In other words, not only matter and electricity but also electric and magnetic fields are

[1] See Sir J. J. Thomson, *Phil. Mag.*, Ser. 6, Vol. XIX., p. 301, February, 1910.

atomic in nature. Accordingly the electron, as the atom of electricity, is to be thought of as a centre on which converge a certain definite number of lines of electric strain, and these lines are in themselves states of strain in the æther, analogous in some sense to vortex filaments in a liquid. To employ a somewhat crude simile, the electron must be thought of as a ball from which proceed in every direction long hairs or filaments radially arranged which it carries about with it.

Sir Joseph Larmor has based an elaborate and consistent theory of electrical phenomena on the supposition that these lines of electric strain radiating from the electron as a centre are lines of torsional strain in the æther. He assumes the æther to be a continuous or extremely fine grained medium, every particle of which resists absolute rotation. This resistance to rotation may proceed from a whirling motion of these very small parts which bestows a gyroscopic stiffness upon the particles. This, however, is not the place to enter upon a discussion of æther theories; the reader may be referred to Sir J. Larmor's book "Æther and Matter" for a description of a working model of this rotational æther based on the well-known properties of the gyroscope. All we shall attempt here is to provide such clear conceptions of the working processes of an electromagnetic field as shall assist the end we have in view.

3. Electric and Magnetic Forces and Fluxes.—The region near electrified bodies, called an electric field, is then the seat of a particular state called *electric strain* which we shall consider is located along certain definite lines called lines of electric strain or sometimes lines of electric force.

Strictly speaking the electric strain is the state in the dielectric caused by an agency called electric force. In the same way the region near magnets or electric currents, called a magnetic field, is the seat of *magnetic flux* located along certain lines called lines of magnetic flux. Electrified bodies and magnetic poles or electric currents exercise attractive or repulsive forces on one another which can be measured in absolute units or *dynes*. The dyne is defined to be the force which, after acting on a mass of 1 gram for 1 second, gives it a velocity of 1 centimetre per second in the direction in which it acts. A unit magnetic pole is one which acts on another unit magnetic pole at a distance of 1 centimetre with a force of 1 dyne. If a unit magnetic pole is placed in a magnetic field the strength of the field or the *magnetic force* at that point is measured by the force in dynes acting on the unit magnetic pole placed there. We shall denote the magnetic force at any point in a field so measured by the letter H. The direction of the lines of magnetic flux in a field can be mapped out by means of iron filings. In the case of a wire carrying a current the lines of magnetic flux are closed lines embracing the wire. The creation of an electric current in a conducting circuit necessitates the existence in it of some source of *electromotive force.* If the conducting circuit is interrupted anywhere, the source of electromotive force still existing in it, a *difference of potential* is created between parts of it, and in the non-conducting region an *electric force* is produced tending to generate electric strain. The presence of an electric field is detected by the existence of a mechanical force acting on a small positively electrified body placed in the field. Two small spheres charged with electricity exert a mechanical force on each other which may be measured in dynes. A unit charge is one which acts on another unit charge at a distance of 1 centimetre with a force of 1 dyne. From a mathematical point of view these electric attractions and repulsions can be regarded as simply the action at a distance of electrons—negative electrons repelling negative and attracting positive and positive repelling positive and attracting negative ones. But as an explanation of what really happens modern scientists do not admit action at a distance, but only the immediate action of contiguous parts of the same medium. Accordingly the forces between electrified bodies must be sought for not in actions at a distance between electrons, but in the immediate actions of their associated lines of electric strain in the universal æther.

It is found that a consistent theory can be built up on the assumption that the lines of electric strain exert a tension like elastic threads and always tend to make themselves as short as possible. Also they exert a lateral pressure, and their arrangement in a field is due to the conflict between their longitudinal tension and lateral pressure.

An explanation of the properties of lines of electric strain is only possible on the basis of some theory of the æther, but it is possible to explain it if we assume a medium possessing inertia and some sort of fine grained whirling structure.

Thus suppose a number of thin inextensible but flexible spherical envelopes or bags to be filled with liquid. If the

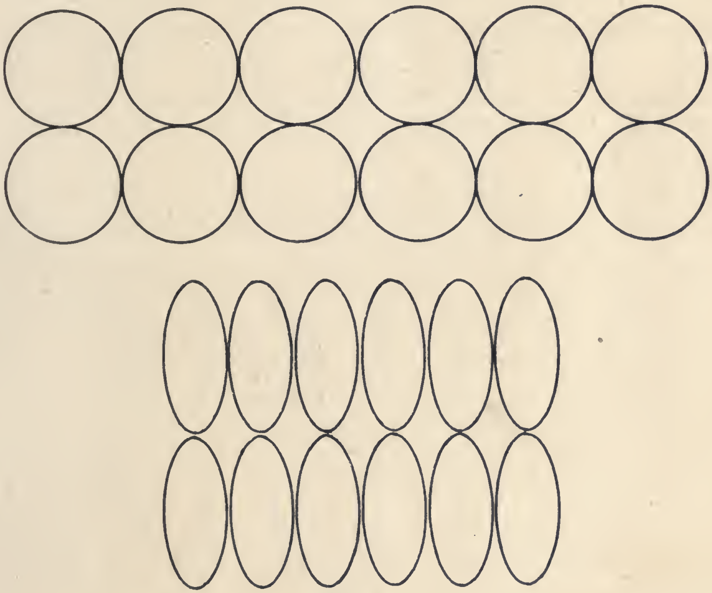

Fig. 2.

liquid in these bags is at rest it will assume a spherical form, but if set in rapid rotation round an axis each spherical ball will become converted into an oblate spheroid like an orange, flattened at the poles and expanded at the equator. If the balls are compelled to remain in contact with each other and if the axes of rotation are arranged in parallel lines, this flattening and expansion of the cells will cause the row of spheres to become shorter along the axis of rotation and also by their equatorial expansion to exert a pressure at right angles, as illustrated in the diagrams in Fig. 2, in which the circles represent the spherical bags which by rotation have become spheroids, thus contracting in length along the line of rotation and expanding laterally.

By some such explanation the student will be able to see that electric attractions and repulsions can be explained by these properties of lines of electric strain. We have to assume that a line of electric strain always starts from a negative electron and ends on a positive one, unless it happens to be self-closed or endless. Furthermore we must assume that in conductors the electrons are quite free to move or that the ends of lines of electric strain can slide along the surface of conductors but cannot so move over the surface of insulators.

We have in the next place to consider the nature of lines of magnetic flux.

Addressing ourselves first to the facts we find that a moving charge of electricity or say an electron creates a magnetic field along circular lines whose planes are perpendicular to its line of motion and centres are on that line. Hence if a spherical charge with radial lines of electric strain moves forward it creates circular lines of magnetic flux embracing its line of motion. The magnetic lines of flux are perpendicular to the directions of the lines of strain and line of motion.

This was first shown experimentally to be the case by H. A. Rowland in 1876 and was confirmed by Rowland and Hutchinson in 1889 and also by Röntgen in 1885. Doubt was thrown on the facts by M. V. Crémieu in 1900, but Rowland's conclusions were reaffirmed by H. Pender in 1901 after a careful research.[1] A brief general description of this classical experiment is as follows :—

A pair of circular glass plates are covered with gold leaf which is divided by radial cuts. These plates are charged to a high potential with electricity and set in rapid rotation round their centres. The two plates are placed parallel and near to each other. Between them is suspended a sensitive shielded magnetic

[1] See H. A. Rowland, *Pogg. Ann.*, 1876, Vol. CLVIII., p. 487 ; Rowland and Hutchinson, *Phil. Mag.*, 1889, Vol. XXVII., p. 445 ; Röntgen, *Ber. der Berlin. Akad.*, 1885, p. 195 ; Crémieu, *Comptes Rendus*, 1900, Vol. CXXX., p. 1544 ; 1901, Vol. CXXXI., pp. 578, 797 ; Vol. CXXXII., pp. 327, 1108 ; H. Pender, *Phil. Mag.*, 1901, Vol. II., p. 179.

needle. When the charged plates revolve at a high speed the needle is deflected in the same manner as it would be if an electric current were flowing round the periphery of the disk. If the plates are charged positively the convection current, as it is called, has the same magnetic effect as a voltaic current flowing round the disk in the direction of rotation and if charged negatively, in the opposite direction.

Hence we have an experimental proof that a moving charge of electricity produces a magnetic field.

It follows that lines of electric strain moving transversely to their own direction create lines of magnetic flux.

A very beautiful direct proof of the fact that a moving charged body is equivalent to an electric current has been given by Professor R. W. Wood.[1] When carbonic dioxide gas strongly compressed in a steel bottle is allowed to escape from a nozzle the sudden expansion creates a fall of temperature sufficient to solidify some of the gas into small particles. These particles of CO_2 are electrified by friction against the nozzle like the particles of water when the steam escapes in Lord Armstrong's hydro-electric machine. The particles of solid carbonic dioxide are electrified positively. If this jet is sent along a glass tube it is possible to obtain velocities of the electrified particles as high as 2,000 feet per second. Professor Wood found that a magnetic needle suspended outside the tube was affected just as if the tube had been a wire conveying an electric current.

In order that we may define more accurately the relation of lines of electric strain and magnetic flux we must attend to the following definitions.

Electric strain may be said to be produced in a dielectric by electric force or stress just in the same manner that mechanical strain is produced by mechanical force or stress. We call the ratio of the stress to the homologous strain the elasticity of the material, and similarly we may call the ratio of the electric stress or force (E) to the electric strain (D) the electric elasticity.

Unfortunately the term dielectric constant (K) or specific inductive capacity was the name given a long time ago to the

[1] See *Phil. Mag.*, 1902, 6th Ser., Vol. II., p. 659.

ratio $\frac{4\pi D}{\mathrm{E}}$. In other words the relation between the total displacement through the surface of a sphere of unit radius at the centre of which is placed a unit charge, to the electric force at a unit of distance has been called the dielectric constant. Suppose that a quantity of electricity Q reckoned in electrostatic units is placed on an extremely small sphere and that we describe round its centre a larger sphere of radius r. Then the surface of this last sphere is $4\pi r^2$ and the displacement per unit of area or number of lines of electric strain passing through this sphere being called D, the total displacement is $4\pi r^2 D$, and this is defined to be equal to Q. Hence the displacement $D = \frac{Q}{4\pi r^2}$.

The electric force E at a distance r is $\frac{Q}{Kr^2}$ where K is the so-called dielectric constant. Hence the ratio of stress to strain is the ratio $\frac{Q}{Kr^2} : \frac{Q}{4\pi r^2} = \frac{4\pi}{K} =$ the electric elasticity and the ratio $4\pi D / E = K =$ the dielectric constant. We do not know the actual number of lines of electric strain proceeding from an electron or natural unit of electricity, but it is convenient to consider that the total number of lines of electric strain proceeding from a charged body is numerically equal to the charge. Thus if the charge is Q there are Q lines of strain passing through the surface of a sphere of radius r described round it. Hence the lines of strain per unit area or the density of the lines, also called the displacement D, is such that $4 \pi r^2 D = Q$.

We have next to consider the relation between the magnetic flux and the electric strain. The magnetic flux (B) is considered to be an effect due to magnetic force (H), and the ratio of the flux to the force is called the magnetic permeability (μ). Hence $B = \mu H$. The magnetic flux density B signifies the number of lines of magnetic force which pass normally through unit of area. Accordingly we have the two fundamental equations of electromagnetism as follows:—

$$B = \mu H \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$D = \frac{K}{4\pi} E \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The occurrence of this 4π in the second equation is due to the mode of definition adopted for the displacement D. It would have been preferable if the electric force E had been so defined that the force at a distance r from a quantity Q were taken as $\frac{Q}{4\pi r^2 K}$, and then this would have given $D = KE$. Taking, however, the usual definition we have the relation as given in the equations above.

We have next to consider the relation between magnetic flux B and electric strain or displacement D. This is based upon the two following facts :—

(i.) That lines of electric strain when moved laterally through a dielectric give rise to lines of magnetic force in a direction at right angles to the lines of electric strain and the direction of their motion.

(ii.) Also that lines of magnetic flux moved laterally through a dielectric give rise to lines of electric strain in a direction at right angles to the lines of flux and to the direction of their motion.

The experimental proof of the first statement has already been given by the experiments of Rowland and others on the magnetic field of moving electric charges.

The second statement when made with regard to a conductor is familiar to us as Faraday's Law of Induction.

If a bar of conducting material of length L is moved perpendicularly to itself with a velocity V across lines of magnetic flux of density B, then we know from Faraday's law that an electromotive force ($E.M.F.$) is set up in the bar such that

$$E.M.F. = BLV$$

reckoned in absolute electromagnetic units or $BLV/10^8$ reckoned in volts.

Now the electric force E is the electromotive force per centimetre of length. Hence $E = E.M.F./L$. Therefore the electric force E set up in the conductor is equal to μHV where H is the magnetic force.

The same will happen if the conductor stands still and the lines of electric strain sweep or cut across it with a velocity V. If the bar is an insulator of dielectric constant K, then it has been shown theoretically by Sir J. Larmor and experimentally by Professor H. A. Wilson that there is an electric force set up in the bar when lines of magnetic force cut across it with a velocity V which is expressed by the equation

$$E=\left(1-\frac{1}{K}\right)\mu HV \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This formula was tested and verified by H. A. Wilson by revolving a cylinder of ebonite at a high speed in a magnetic field the lines of which were parallel to the axis of the cylinder around which it revolved. The difference of potential between the axis and perimeter was measured and the mean electric force equal to the above difference of potential divided by the radius of the cylinder was calculated and found to agree with the above formula. For details the reader is referred to the original paper (see *Philosophical Transactions* of the Royal Society of London, Vol. 204A, p. 121, 1905; also *Proc. Roy. Soc.*, Vol. 73, p. 490, 1904).

As regards the magnetic force produced by the lateral movement of a line of electric strain, it can be shown that if E is the electric force in the direction of the lines of electric strain and if K is the dielectric constant of the medium, and if V is the velocity of the lines parallel to themselves, then the magnetic force H produced by the motion is given by the formula

$$H=KEV \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Otherwise, if D is the displacement or number of lines of electric strain passing through unit area, and if they move with a velocity V in a direction inclined at an angle θ to the direction of the lines of strain, then the magnetic force H due to their motion is given by

$$H=4\pi DV \operatorname{Sin}\theta \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A statement of the connection between the electric force E and the magnetic force H can be arrived at in another way. Suppose we describe any small area in an electric field, say a rectangle of which the sides are dx and dy, and let the electric force E at the centre of that area have rectangular components E_x and E_y parallel to dx and dy respectively. Imagine that we travel round the area in a counter-clockwise direction, multiplying the length of each side by the component of the electric force in its direction and reckoning the product as positive when the force is in the direction of motion and negative when it is against it, and finally add up algebraically all these products, we obtain what is called the *line integral of the force* round the area. Thus for the case in question we have for the line integral the sum

$$\left(E_x-\frac{1}{2}\frac{d\,E_x}{dy}\,\delta y\right)\delta x+\left(E_y+\frac{1}{2}\frac{d\,E_y}{dx}\,\delta x\right)\delta y$$
$$-\left(E_x+\frac{1}{2}\frac{d\,E_x}{dy}\,\delta y\right)\delta x-\left(E_y-\frac{1}{2}\frac{d\,E_y}{dx}\,\delta x\right)\delta y$$
$$=\left(\frac{d\,E_y}{dx}-\frac{d\,E_x}{dy}\right)\delta x\,\delta y \quad . \quad . \quad . \quad . \quad (9)$$

The above line integral is the electromotive force acting round the area, and the quantity in the brackets, viz., $\frac{d\,E_y}{dx}-\frac{d\,E_x}{dy}$ is called the *curl* of the electric force at that point and written Curl E. If there is a magnetic force H in a direction z at right angles to the plane of xy, then the total magnetic flux through the area $\delta x\,\delta y$ or the number of lines of electric force passing through the area is $\mu H\,\delta x\,\delta y$ where μ is the permeability. If then the electromotive force is due to the variation of this field we have by Faraday's law

$$\left(\frac{d\,E_y}{dx}-\frac{d\,E_x}{dy}\right)\delta x\,\delta y=-\frac{d}{dt}\left(\mu H\,\delta x\;\delta y\right) \quad . \quad . \quad (10)$$

or

$$\text{Curl}\,E=-\mu\,\dot{H} \quad . \quad . \quad . \quad . \quad (11)$$

where $\dot{H}$ stands for $\frac{d\,H}{dt}$ or the time variation of H. For the above formula merely expresses the fact that the electromotive force is due to the time rate of change of the magnetic flux through the area.

Again, if H is the magnetic force in any field and if its rectangular components are H_x and H_y the quantity $\frac{d\,H_y}{dx}-\frac{d\,H_x}{dy}$ formed in the same manner as in the case of the electric force is called the Curl of the magnetic force. If then D is the electric displacement normally through the area $\delta x\,\delta y$ drawn in the magnetic field, the time rate of change of this displacement denoted by $\frac{d\,D}{dt}$ or $\dot{D}$ is called the dielectric current and is the rate at which electricity is moved through the area. According to Maxwell's theory this dielectric current produces magnetic force according to the same laws as a current of conduction. Hence 4π times the total current through the area is equal to the line integral of magnetic force round the area. Applying this to the above case of the dielectric current through the area $\delta x\ \delta y$ we have

$$\left(\frac{d\,H_y}{dx}-\frac{d\,H_x}{dy}\right)\delta x\ \delta y=4\pi\frac{d\,D}{dt}\delta x\ \delta y$$

or
$$\text{Curl } H=4\pi\,\dot{D}$$

or
$$\text{Curl } H=K\dot{E} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The expressions therefore for the Curl of the magnetic force and for the Curl of the electric force are quite similar and involve the two constants of the dielectric, viz., the magnetic permeability μ and the dielectric constant K.

It can be shown that the velocity of propagation of any electromagnetic disturbance or state through a dielectric is equal to $1/\sqrt{K\mu}$.

For if we consider that E and H are both at right angles to a common direction taken as the x-axis and vary in that direction alone, that is are propagated in that direction, we have for the Curl equations

$$\frac{d\,H}{dx}=-K\frac{d\,E}{dt} \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{d\,E}{dx}=-\mu\frac{d\,H}{dt} \quad . \quad . \quad . \quad . \quad (14)$$

Hence differentiating with regard to x and t we can easily find that

$$\frac{d^2\,H}{dx^2}=K\mu\frac{d^2\,H}{dt^2} \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{d^2\,E}{dx^2}=K\mu\frac{d^2\,E}{dt^2} \quad . \quad . \quad . \quad . \quad (16)$$

Now these equations are precisely similar in form to those we deduced for the velocity of sound (see Equation (1)), and they show that the velocity of an electromagnetic disturbance spreads through the dielectric with a velocity u such that $u=\dfrac{1}{\sqrt{K\mu}}$.

Thus if we suppose a current in a conductor buried in a dielectric to be suddenly reversed in direction, the magnetic field due to it is not reversed in direction everywhere at once, but the reversal begins at the surface of the conductor and travels outwards with a velocity $1/\sqrt{K\mu}$ where K and μ are the electric and magnetic constants of the dielectric. As regards numerical values we do not know the separate absolute values of K and μ for air or empty space, that is for the æther, but we do know that the value of the velocity u is very nearly 3×10^{10} cms. per second or about 1,000 million feet a second—that is the velocity of light.

Accordingly, if lines of electric strain are created at one point in a dielectric they diffuse or travel through it with a velocity u called the electromagnetic velocity, and as they move they give rise to lines of magnetic flux at right angles to themselves and to their direction of motion. If E is the electric force and K the dielectric constant, then the magnetic force H resulting from the sidewise motion of the lines of electric strain is given by

$$H=KEu \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Also if lines of magnetic flux move in a similar manner the electric force E created is given by

$$E=\mu Hu \quad . \quad . \quad . \quad . \quad . \quad (18)$$

4. Electromagnetic Waves along Wires.—We are now in a position to explain more in detail the nature of an electromagnetic wave.

As we are not concerned here with electric waves in space or so-called free or Hertzian waves, but only with waves guided along wires, we shall take a concrete case, viz., a pair of long parallel wires of very good conducting material, and examine the effects taking place when an electromotive force of particular type is applied between them.

Let us suppose an alternator to be applied at one end giving an electromotive force which rises suddenly to a certain value, maintains it constant for a while, then vanishes and is shortly afterwards replaced by a reversed electromotive force going through the same cycle of values. The curve of electromotive force or the variations of *E.M.F.* with time would then be represented by a square-shouldered curve as in Fig. 3.

If then the *E.M.F.* rises suddenly at one end of the pair of wires it implies that there is an electric force and therefore an electric strain in the space between. Looking at the wires end on, the strain would be distributed in curved lines as in Fig. 4,

FIG. 3.

where the small circles marked with a dot and a cross represent the section of the wires. When looked at from the side the lines of electric strain would project into straight lines as in Fig. 5, in which the arrow heads represent the direction of the electric strain. Now this strain does not make its appearance at all distances at once, but is propagated outwards in the space between and around the wires at a certain speed, and when the electromotive force at the sending end dies down suddenly it does not cease at all points at once. The effect is equivalent to a gradual movement of lines of strain along the space between the wires. This movement implies movement of electric charges along the wires. The ends of the lines of electric strain, so to speak, slip along the wires, and we may regard their ends as terminating on electric charges. But this lateral movement of lines of electric strain and of longitudinal movement of electric charges implies the flow of

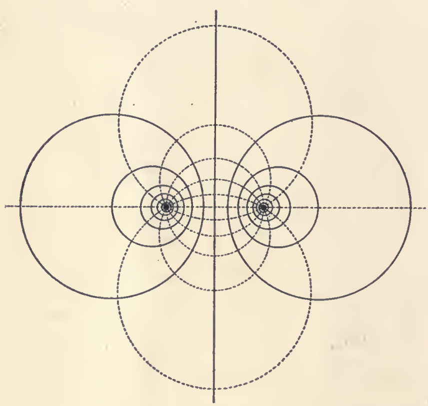

FIG. 4.—End-on view of Lines of Electric and Magnetic Force of parallel wires. Firm lines are Magnetic lines, Dotted lines are Electric lines.

an electric current along the wires and the creation of lines of magnetic flux in the interspace, which lines of flux are everywhere perpendicular to the lines of electric strain and the direction of the motion of the latter. The lines of flux are therefore closed loops embracing the wires as shown by the dotted lines in Fig. 4, and their section is represented by the dots in Fig. 5. The two distributions of lines of strain and flux travel together, and they both represent energy in different forms. If the electric strain density or number of lines of electric strain per square centimetre is represented by D and the number of lines of magnetic flux per square centimetre is represented by B, and if the dielectric constant is K and the magnetic permeability is μ, then

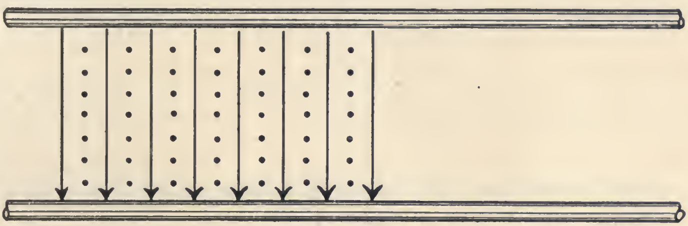

FIG. 5.—Sidewise view of Lines of Electric and Magnetic Force of parallel wires. The arrows are electric lines and dots the magnetic lines.

the energy of electric strain per cubic centimetre is represented by $\frac{1}{2}DE = \frac{1}{2}\frac{D^2}{K}$, and the energy of magnetic flux per cubic centimetre by $\frac{1}{2}HB = \frac{1}{2}\frac{B^2}{\mu}$, provided that the flux and strain lines are respectively practically parallel through the cubic centimetre, and when $B = \mu H$ and $D = KE$. If, however, the values of the electric and magnetic forces created by the motion are controlled by the relations $H = KEu$ and $E = \mu Hu$, then it follows that

$$\frac{1}{2}\frac{B^2}{\mu} = \frac{1}{2}\frac{D^2}{K} \text{ or } \frac{1}{2}KE^2 = \frac{1}{2}\mu H^2$$

In other words, the total energy is equally divided between electric and magnetic forms.

Hence as soon as the lines of electric strain begin to move freely they have to part with some energy or some have to go out of existence to create the lines of magnetic strain, and the total energy is equally divided between the two sets of lines.

If the lines stop moving, then the magnetic flux lines vanish, but their energy cannot simply disappear, but it is conserved and reappears as the energy of additional lines of electric strain created. Conversely if the lines of electric strain disappear, then their energy is transmitted into additional lines of magnetic flux.

Such a block or group of lines of electric strain travelling through a dielectric with associated lines of magnetic flux at right angles to the lines of strain, the two groups being of equal

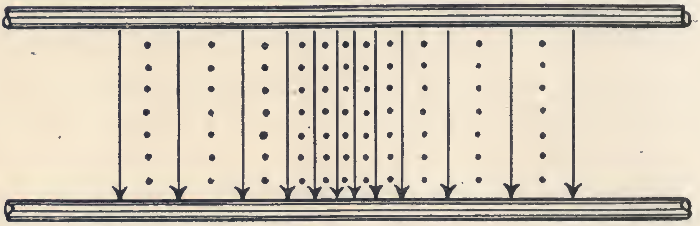

FIG. 6.

energy and mutually sustained by their motion, is called an electromagnetic wave.

Generally speaking, in an electromagnetic wave the electric lines or force do not begin and end sharply, but fade away fore and aft in accordance with a sine law of variation, so that it may be diagrammatically represented as in Fig. 6, where the closeness of the lines is supposed to denote the electric force and of the dots the magnetic force. We may then mathematically express the electric strain and magnetic flux symbolically as follows:—

$$D = D_0 \text{ Sin } (x - Vt) \quad . \quad . \quad . \quad . \quad (19)$$

$$B = B_0 \text{ Sin } (x - Vt) \quad . \quad . \quad . \quad . \quad (20)$$

where D_0 and B_0 represent the maximum values of the electric strain and magnetic flux and D and B their values at any distance x from an origin and any time t, and V is the velocity of propagation. For these expressions are periodic both in space and time and remain the same if for x we put $x + \lambda$ and for t, $t + T$, provided $\lambda/T = V$. The length λ is called the wave length and the time T is called the periodic time. The former is the length in which the whole cycle or series of electric or magnetic lines is contained at any instant, and the time T is the time in which the whole cycle of variations completes itself at any one place. If then we have an ordinary alternator attached to the end of the line, producing a simple harmonic electromotive force, we have sinoidal electromagnetic waves travelling up the space between the wires with a velocity V presently to be determined and constituting a train of electromagnetic waves. In a pure electromagnetic wave the energy is half electric and half magnetic, and the two constituents, the magnetic component and the electric component, travel together with the same speed and are in step as regards phase. As regards the relative direction of the electric force or strain, magnetic force or flux, and motion, their direction can be remembered by holding the thumb, middle finger, and fore finger of the right hand in directions as nearly as possible at right angles. Let the direction in which the thumb points indicate the direction of the wave motion or velocity, the direction in which the middle finger points the direction of the magnetic force or flux, and the direction in which the fore finger points the direction of the electric force or strain. Then by twisting round the hand into various directions with the thumb, and two fingers held stiffly at right angles, we can always determine the directions of the magnetic and electric vectors, as they are termed, with regard to the direction of wave propagation.

5. Reflection of Electromagnetic Waves at the End of a Line.—Before proceeding to discuss analytically the propagation of waves along wires it will be found profitable to consider the phenomena which occur when an electromagnetic wave reaches the end of a line whether open or closed.

First consider an open or insulated end. When the lines of strain arrive at the end of the line they cannot proceed farther because their ends cannot be detached from the metal wires, but their inertia causes them to travel as far as they can by stretching themselves; hence as they reach the end of the line they extend themselves in curved lines as shown in Fig. 7. As soon, however, as they come to rest, the accompanying magnetic flux, which is produced only by the sidewise motion of the strain lines, vanishes, but its disappearance results in the creation of additional lines of electric strain to conserve the

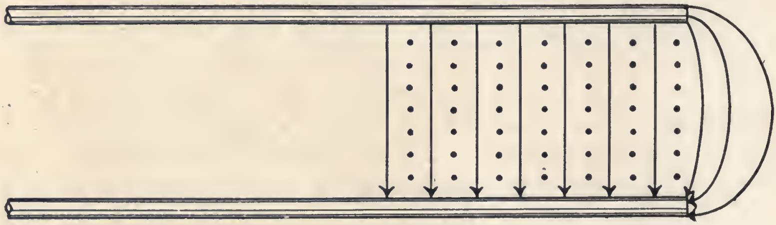

FIG. 7.

energy. Some of the electric strain lines are then in a state of stretch, but owing to their longitudinal tension they tend to contract and to start the whole mass of strain lines back again on the return journey. As soon, however, as the lines begin to travel their motion recreates the magnetic flux lines, and part of the electric strain lines vanish to supply the magnetic energy. Then the wave is re-established and runs back again to the origin. Here it may be reflected again and so travel backwards and forwards until its energy is dissipated. If the receiving end

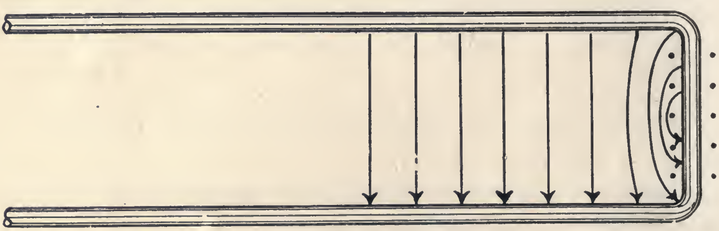

FIG. 8.

of the cable is short circuited by a good conductor, then the process of reflection is somewhat different. When the strain lines arrive at the end their ends follow on round the short circuit, and the strain lines therefore tend to shrivel up to nothing, as shown in Fig. 8. But this process implies a movement of each part of the strain line at right angles to itself and so gives rise to a line of magnetic flux embracing the end conductor, which is left behind as the equivalent form of energy when the electric strain lines disappear. Hence when the wave reaches the closed circuit end all the lines of electric strain vanish for the moment and are replaced by lines of magnetic flux. But this state is not stable. The closed lines of magnetic flux closely embracing the short circuit end begin to expand outwards again like ripples on a pond, and the moment they move their motion recreates electric strain lines, and presently the energy is again divided equally between electric strain and magnetic flux lines in lateral motion.

Accordingly we see that there are two general laws as follows.

1. When an electromagnetic wave is reflected at the open end of a cable the magnetic component is reversed in direction, and at the moment of reflection the magnetic component is suppressed and the electric component doubled in intensity.

2. When an electromagnetic wave is reflected at the short circuited end of a cable the electric component is reversed on reflection, and at the moment of reflection the electric component is suppressed and the magnetic component doubled in intensity. If the electromagnetic waves continue to arrive and to be reflected at the open or closed end, then the two trains of waves, direct and reflected, pass through each other, and the resultant state of affairs is said to be due to the interference of the direct and reflected wave trains.

If the receiving end is not perfectly insulated or perfectly short circuited the energy of the wave is partly reflected and partly transmitted and the resulting condition becomes still more complicated.

We may make an additional inference. If there be in any cable a part in which there is greater inductance or capacity per unit of length than at other parts, these lumps of capacity or inductance will cause partial reflection of the wave.

The whole process of transmission and reflection of electromagnetic waves up the space between two conducting wires is exactly analogous to the phenomena occurring when air waves are travelling up a pipe such as an organ pipe.

In place of magnetic flux we have to consider the velocity of the air particles, and in place of electric strain we have air condensation or rarefaction. If the pipe is closed at the end then when the air wave reaches it, it is reflected with change of the direction of the velocity component. If the pipe is open at the end the wave is also reflected, but with change of phase of the density component, that is a condensation is reflected as a rarefaction and *vice versâ*.

In the air wave at any one point changes of density succeed each other periodically, and also changes in the velocity of the air particles. In the electromagnetic wave changes of electric strain and electric force in amount and direction succeed each other at any one point, and also similar changes in magnetic flux or force.

If the wave could be arrested and fixed in its state at any one moment we should find a periodic distribution of electric and magnetic force in space, the two being in mutually perpendicular directions and also at right angles to the direction of propagation.

6. Differential Equations expressing the Propagation of an Electromagnetic Disturbance along a pair of Wires.—Having obtained a general conception of the nature of the physical processes taking place when a simple periodic electromotive force is applied to a pair of parallel wires, we shall next proceed to translate these ideas into mathematical language in order to give greater precision to them.

Let us consider a transmission line consisting of two parallel infinitely long wires having a resistance R ohms per mile of line, that is per mile of lead and return, and a capacity C farads per mile, an inductance L henrys per mile, and a dielectric conductance of S mhos per mile, the mho being the reciprocal of the ohm.

Let v be the potential difference between the wires at any distance x from the sending end and let i be the current at that point. Then the potential difference and current at a distance $x+\delta x$ is $v+\frac{dv}{dx}\delta x$ and $i+\frac{di}{dx}\delta x$.

The potential difference ($P. D.$) is partly expended in driving the current against the ohmic resistance and partly in overcoming the back electromotive force due to inductance. Hence for a length δx we have the following equations.

$$\frac{dv}{dx}\delta x = R\delta x i + L\delta x\frac{di}{dt} \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{di}{dx}\delta x = S\delta x v + C\delta x\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad (22)$$

The first equation expresses the manner in which the fall in voltage down the length δx is accounted for, and the second the manner in which the current in the same length is expended partly in charging the wire and partly in conduction across the dielectric. From these equations we at once derive the following:—

$$\frac{dv}{dx} = Ri + L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad (23)$$

$$\frac{di}{dx} = Sv + C\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad (24)$$

These are the differential equations for the propagation of a current in a line having resistance, inductance, capacity, and insulation conductance. We need not consider at the present moment the general solution of these equations, but for the immediate purposes we shall limit our consideration to the case in which both v and i are simple sine functions of the time. Then if $i = I \text{ Sin } pt$ and $v = V \text{ Sin } (pt + \theta)$, these functions indicate a simple sine variation of i and v with a difference of phase θ but equal frequency n such that $2\pi n = p$. Thus we have $\frac{di}{dt} = p\,I \text{ Cos } pt$, $\frac{dv}{dt} = p\,V \text{ Cos } (pt+\theta)$. Hence if we denote the periodic current by a simple vector representing its maximum value, then a vector p times as long at right angles to the vector denoting the current will represent the maximum value of the time rate of change of the current or the maximum value of $\frac{di}{dt}$.

If therefore any line is taken to represent RI, or the maximum value of Ri, then for the maximum value of $L\frac{di}{dt}$ we must draw a line to the same scale representing LpI at right angles to the vector RI. The vector sum of these lines or $RI + jpLI$ will be a line representing the maximum value of $\frac{dV}{dx}$. Hence when the time variation of i and v are simple harmonic, we can, in place of the scalar equation

$$\frac{dv}{dx}=Ri+L\frac{di}{dt},$$

write the vector equation

$$\frac{dV}{dx}=RI+jpLI,$$

where V and I are the maximum values of v and i during the period. We thus eliminate the time variable and deal only with the maximum values of the quantities during the period.

Hence we can write our two fundamental equations in the form,

$$\frac{dV}{dx}=(R+jpL)\ I \quad \ldots \quad (25)$$

$\omega = 2\pi f$

$$\frac{dI}{dx}=(S+jpC)\ V \quad \ldots \quad (26)$$

The quantity, $R + jpL$ is called the *vector impedance*, and the quantity $S + jpC$ is called the *vector admittance*.

By differentiating each of the equations above with regard to x we can separate the variables and arrive at the two equations,

$$\frac{d^2V}{dx^2}=(R+jpL)\ (S+jpC)\ V=P^2\ V \quad \ldots \quad (27)$$

$$\frac{d^2I}{dx^2}=(R+jpL)\ (S+jpC)\ I=P^2\ I \quad \ldots \quad (28)$$

where $$P=\sqrt{R+jpL}\ \sqrt{S+jpC}=\alpha+j\beta \quad \ldots \quad (29)$$

P is a complex quantity and therefore may be written in the form $\alpha + j\beta$.

It is called the *Propagation constant* of the line.

Squaring the two sides of the expression (29) we have

$$\alpha^2-\beta^2+j\ 2\ \alpha\beta=(RS-p^2\ LC)+j\ (p\ LS+p\ CR),$$

and equating horizontal and vertical steps we have,

$$\alpha^2-\beta^2=RS-p^2\ LC \quad \ldots \quad (30)$$

and $$2\alpha\beta=p(LS+CR) \quad \ldots \quad (31)$$

and equating the sizes of the vectors we have

$$\sqrt{\alpha^2+\beta^2}=\sqrt{\sqrt{R^2+p^2L^2}\,.\,\sqrt{S^2+p^2C^2}}$$

or,

$$\alpha^2+\beta^2=\sqrt{(R^2+p^2L^2)\,.\,(S^2+p^2C^2)} \quad . \quad . \quad (32)$$

whence we find that

$$\alpha=\sqrt{\frac{1}{2}\left\{\sqrt{(R^2+p^2L^2)(S^2+p^2C^2)}+(SR-p^2LC)\right\}} \quad . \quad (33)$$

$$\beta=\sqrt{\frac{1}{2}\left\{\sqrt{(R^2+p^2L^2)(S^2+p^2C^2)}-(SR-p^2LC)\right\}} \quad . \quad (34)$$

These quantities α and β are very important. α is called the *attenuation constant*, and β is called the *wave length constant*, and $P=\alpha+j\beta$ is the *Propagation constant* of the line.

The expressions for α and β may be modified by relative values of R, L, S and C, which last are called the *primary constants* of the line, α, β, and P being the *secondary constants.*

Thus if $S=0$ or the line has no leakance, then

$$\alpha=\sqrt{\frac{1}{2}Cp\left\{\sqrt{R^2+p^2L^2}-Lp\right\}} \quad . \quad . \quad . \quad (35)$$

$$\beta=\sqrt{\frac{1}{2}Cp\left\{\sqrt{R^2+p^2L^2}+Lp\right\}} \quad . \quad . \quad . \quad (36)$$

If $S=0$ and $L=0$ or the line has no inductance as well, then

$$\alpha=\beta=\sqrt{\frac{CpR}{2}} \quad . \quad . \quad . \quad . \quad (37)$$

In all these cases α and β are functions of p and therefore of the frequency n, since $p=2\pi n$.

In the general expressions for α and β, viz., in equations (33) and (34), if we add and subtract the quantity $2p^2CLSR$ to and from the product $(R^2+p^2L^2)\,(S^2+p^2C^2)$ we can throw the expressions for α and β in the form

$$\alpha=\sqrt{\frac{1}{2}\left\{\sqrt{(SR+p^2LC)^2+p^2(LS-CR)^2}+(SR-p^2LC)\right\}} \quad . \quad (38)$$

$$\beta=\sqrt{\frac{1}{2}\left\{\sqrt{(SR+p^2LC)^2+p^2(LS-CR)^2}-(SR-p^2LC)\right\}} \quad . \quad (39)$$

If then the primary constants have such values that

$$LS-CR=0 \text{ or } LS=CR,$$

then we have

$$\alpha=\sqrt{SR} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\beta=p\sqrt{CL} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

For a reason to be explained later on, such a cable is called a *distorsionless cable*, and in that case the value of a is the same for all frequencies.

We have therefore obtained differential equations expressing the relations of the potential and current at any point in a line under the assumption that the current and potential are quantities which vary in accordance with a simple sine law.

In the next chapter we shall discuss the solution of these simple periodic equations in various cases, and deal with the general solution of the differential equations (23) and (24) at a later stage.

CHAPTER III

THE PROPAGATION OF SIMPLE PERIODIC ELECTRIC CURRENTS IN TELEPHONE CABLES

1. The Case of an Infinitely Long Cable with Simple Periodic Electromotive Force at the Sending End.—Returning to the fundamental differential equations we have now to find solutions for particular cases. These equations are,

$$\frac{d^2V}{dx^2}=P^2V \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d^2I}{dx^2}=P^2I \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where V and I are the maximum values during the period of the potential and current at any point in the line. A differential equation of this type can be satisfied by simple exponential solutions of the form $V=A\epsilon^{+Px}$ and $V=B\epsilon^{-Px}$ where A and B are constants, as can be seen at once by double differentiation of these last expressions. Hence a solution of these equations is found by taking the sum of the above particular solutions, viz.,

$$V=A\epsilon^{+Px}+B\epsilon^{-Px} \quad . \quad . \quad . \quad . \quad (3)$$

$$I=C\epsilon^{+Px}+D\epsilon^{-Px} \quad . \quad . \quad . \quad . \quad (4)$$

where A, B, C, and D are constants to be determined.

In obtaining the original differential equations (23) and (24) (see Chapter II.) it is to be noticed that we assumed the current and potential to *increase* with x. It is most convenient to reckon the distance x from the sending end, and then V and I both diminish with x. We can, however, make the necessary change in our solutions by writing $-x$ for x. Making the change, we have for the solutions of (1) and (2).

$$V=A\epsilon^{-Px}+B\epsilon^{+Px} \quad . \quad . \quad . \quad . \quad (5)$$

$$I=C\epsilon^{-Px}+D\epsilon^{+Px} \quad . \quad . \quad . \quad . \quad (6)$$

Suppose that the cable is infinitely long, then when $x = \infty$ we must have $V = 0$ and $I = 0$, and therefore the constants B and D must both be zero. Again when $x = 0$ we must have $V = E_0$ where E_0 is the potential difference of the two members of the cable at the origin, or, in other words, the electromotive force applied at the sending end. Hence $A = E_0$. Moreover, since $-\frac{dV}{dx} = (R + jp\,L)\,I$, it follows that

$$PE_0\epsilon^{-Px} = (R+jpL)C\epsilon^{-Px} \text{ or } C = \frac{E_0 P}{R+jpL}.$$

Hence in the case of the infinitely long cable with simple periodic $E.M.F.$ E_0 applied at the sending end, the potential and current at any distance x are given by the equations,

$$V = E_0\epsilon^{-Px} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$I = E_0\frac{\sqrt{S+jpC}}{\sqrt{R+jpL}}\epsilon^{-Px} \quad . \quad . \quad . \quad . \quad (8)$$

where $$P = \sqrt{R+jpL}\ \sqrt{S+jpC} = \alpha + j\beta.$$

The quantity P is called the *propagation constant* and Px is called the *propagation* length or *distance*.

The quantity $\frac{\sqrt{R+jpL}}{\sqrt{S+jpC}}$ is of great importance and is called the *Initial sending end Impedance* and denoted by Z_0.

Bearing in mind that $\epsilon^{-j\beta x} = \text{Cos}\ \beta x - j\ \text{Sin}\ \beta x$ and that $\epsilon^{-Px} = \epsilon^{-\alpha x - j\beta x}$ it is seen that the solutions of the differential equations for the case of the infinitely long cable can be put in the form

$$V = E_0\,\epsilon^{-\alpha x}\,(\text{Cos}\ \beta x - j\ \text{Sin}\ \beta x) \quad . \quad . \quad . \quad (9)$$

$$I = \frac{E_0}{Z_0}\,\epsilon^{-\alpha x}\,(\text{Cos}\ \beta x - j\ \text{Sin}\ \beta x) \quad . \quad . \quad . \quad (10)$$

Each of these expressions is a complex quantity and therefore represents a vector. The value of V is obtained by operating on E_0 with two factors; one viz., $\epsilon^{-\alpha x}$, called the *attenuation factor*, continually decreases in a geometric progression as x increases in an arithmetic progression. The other factor $(\text{Cos}\ \beta x - j\ \text{Sin}\ \beta x)$, called the *phase factor*, repeats itself over and over again in value at intervals of distance equal to $\frac{2\pi}{\beta}$ as x continually increases, since $\text{Cos}\ \beta x = \text{Cos}\ \beta\left(x+\frac{2\pi}{\beta}\right)$ and the same for the sine term.

It is clear, therefore, that as we move along the line the potential and current rise and fall periodically, but so that the maximum value in each space period dies gradually away. Moreover at each point the potential and current are periodic with time; that is, run through a cycle of values.

This shows that as we proceed along the cable, taking the potential and current at each point to be the maximum values they have during the period, we find that these maximum values attenuate in a certain ratio and are shifted backwards in phase relatively to each other. At equal space intervals along the line

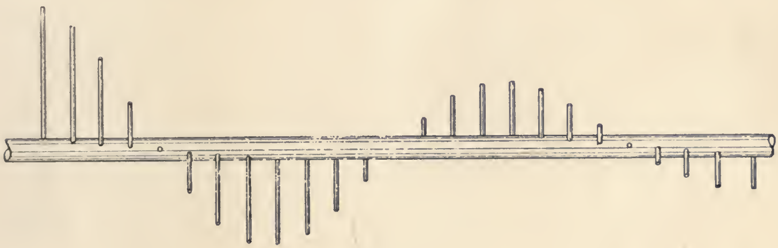

FIG. 1.

these maximum values form a geometric series as regards their size and their phases differ by equal angles. The distance $\frac{2\pi}{\beta}$ is called the *wave length* and denoted by λ.

We can represent the state of affairs in the cable by a model made in the following manner :

Take a long wooden rod to represent the cable and a number of wires the lengths of which form a geometric series, that is the length of each wire is the same fraction or percentage of the next longest one. Let holes be bored in the wooden rod at equal distances and in such directions that these holes lie on a spiral of equal pitch wound round the rod, the holes being otherwise perpendicular to the axis.

Then if the wires are inserted into the holes we shall have a structure as shown in Fig. 1.

Each wire will then represent in magnitude and direction the maximum value and phase of the current or potential at the corresponding point in the cable. If the tips of all these wires are joined by another wire, this last will form a spiral round the rod, but the spiral will be like a corkscrew, decreasing in diameter the further we move along the rod. If we wish to represent the changes which take place from instant to instant in the potential or current we must place this rod in the sunshine and cast the shadow of it on a sheet of white paper held perpendicular to the sun's rays. If then the rod is rotated the shadow of each of the wires will increase and decrease and reverse direction at each turn. The length of the shadow at any instant will denote the actual current or potential at that point in the cable and runs through a cycle of values at each revolution of the rod.

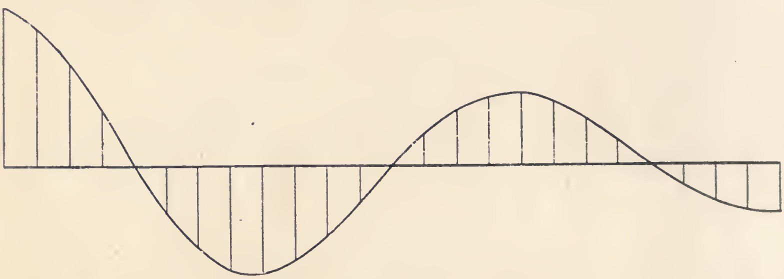

Fig. 2.

A line joining the tips of all the shadows will at any moment be a wavy decrescent curve as in Fig. 2, and as the rod is rotated the ends of these shadow lines will appear to move forward with a wavy motion.

The curve formed by joining the tips of the shadow lines is a curve like that in Fig. 2 whose equation is of the form

$$y = A\,\epsilon^{-\alpha x} \operatorname{Cos} \beta x \quad . \quad . \quad . \quad . \quad (11)$$

Hence if we suppose ourselves to stay permanently at a point in the cable the distance of which from the sending end is x, we should find the potential and current at that point varying periodically with a frequency n or having a periodic time T.

If we could cause the current and potential at all points in the line to be fixed permanently in the state in which they are at any instant t, then we should find a distribution along the line which is periodic with a wave length $2\,\pi/\beta$, but the maximum values in each half wave length decreasing in the ratio $\epsilon^{-\alpha\frac{\pi}{\beta}}$.

A model imitating the actual changes of potential from instant to instant at any point in the cable can be made in the following manner.

On a long axis are fixed a series of grooved eccentric wheels the eccentricities of which decrease in geometric progression—that is the eccentricity of each one is the same fraction of that of the preceding one all the way along. Also the *angle of lead* of these wheels decreases progressively by equal angular steps. In each wheel is a groove on which is hung a long loop of string carrying a weight at the bottom. The loops are all of equal length. These weights therefore are arranged along a wavy decrescent curve. If the axis is rotated each bob moves up and down with a nearly simple harmonic motion, but the amplitudes of motion decay in a geometric progression and the phases lag in arithmetic progression, and hence the bob motion represents in phase and amplitude the potential or current at various points along the cable.

A model of this kind has actually been constructed by the author and exhibited in various places.[1]

If then for any cable we are given the primary constants R, L, C, S, in ohms, henrys, farads, and mhos, per mile, we can calculate the values of the attenuation constant α and the wave length constant β and hence the attenuation factor $\epsilon^{-\alpha x}$ and the phase factor $\text{Cos}\ \beta x - j\ \text{Sin}\ \beta x$ for any distance x. The attenuation per mile, viz., $\epsilon^{-\alpha}$, and the wave length $2\pi/\beta$ are then at once found.

The value of $\epsilon^{-\alpha x}$ can be calculated most easily by means of a Table of Hyperbolic Sines and Cosines.

For $$\epsilon^{-\alpha x} = \text{Cosh}\ \alpha x - \text{Sinh}\ \alpha x.$$

Hence $$V = E_0\,(\text{Cosh}\ \alpha x - \text{Sinh}\ \alpha x)(\text{Cos}\ \beta x - j\ \text{Sin}\ \beta x) \quad . \quad (12)$$

$$I = \frac{E_0}{Z_0}\,(\text{Cosh}\ \alpha x - \text{Sinh}\ \alpha x)(\text{Cos}\ \beta x - j\ \text{Sin}\ \beta x) \quad . \quad (13)$$

[1] See "A Model illustrating the Propagation of a Periodic Current in a Telephone Cable and the Simple Theory of its operation," *Phil. Mag.*, August, 1904, and *Proc. Phys. Soc. Lond.*, Vol. XIX.

If we reckon phase angles from the direction of E_0, then symbolically we have

$$V=(E_0)(\mathrm{Cosh}\ \alpha x-\mathrm{Sinh}\ \alpha x)\overline{\diagdown \beta x} \quad . \quad . \quad . \quad (14)$$

$$I=\left(\frac{E_0}{Z_0}\right)(\mathrm{Cosh}\ \alpha x-\mathrm{Sinh}\ \alpha x)\overline{\diagdown \beta x} \quad . \quad . \quad . \quad (15)$$

where the brackets round E_0 and $\frac{E_0}{Z_0}$ denote the sizes of these vectors.

We have therefore completely determined the potential and current at any point in the infinite cable.

Moreover, given the values of R, L, C, and S, per mile or per kilometre, we can calculate the value of the attenuation constant α and hence of $\epsilon^{-\alpha}$, which gives us the attenuation per mile or ratio in which the maximum values of the current and potential are weakened by going a mile or kilometre along the cable. Also we can calculate the value of the wave length constant, which the formula (34), Chapter II., gives in *radians*, the radian being the unit angle or angle whose arc is equal to the radius, viz., $\frac{180}{\pi}=57^\circ\ 17'\ 45''$ nearly. Accordingly the angle of the vector denoting the current or potential is shifted backwards by $\beta\,\frac{180}{\pi}$ degrees per mile or per kilometre.

Hence after running a distance $\frac{2\pi}{\beta}$ the phase has shifted backwards 360° and the cycle as regards phase begins again to be repeated. The length $2\pi/\beta=\lambda$ is called the wave length.

Now in all cases of wave motion the wave velocity W is connected with the wave length λ and the frequency n in the relation given by the formula

$= \frac{2\pi}{\lambda}$

$$W=n\lambda \quad . \quad . \quad . \quad . \quad . \quad (16)$$

But $\lambda=2\pi/\beta$ and $2\pi n=p$, and hence

$= \frac{w}{V_\lambda}$

$$W=\frac{p}{\beta} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Accordingly the velocity of the wave is a function of the frequency n.

It is therefore seen that in an ordinary cable alternating currents or potentials of different frequency decay at different rates along the cable and travel with different velocities. There is, however, one important case in which currents of all frequencies attenuate equally and travel at the same speed. This is when the primary constants have such values that

$$\frac{R}{L}=\frac{S}{C} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

We have seen that under these conditions

$$\alpha=\sqrt{SR} \text{ and } \beta=p\sqrt{CL}.$$

Hence $W=\frac{1}{\sqrt{CL}}$. When this is the case both α and W are independent of the frequency, and currents and potentials of all frequencies travel and attenuate alike.

Such a cable has been called by Mr. Oliver Heaviside a distorsionless cable, for reasons to be considered later on.

In the case of all ordinary cables the values of the constants are such that the product $R\,C$ is much greater than the product $L\,S$. It is easily seen that under these conditions the lower the frequency the less the attenuation and wave velocity but the greater the wave length. Hence shorter waves travel faster and attenuate more rapidly.

Thus for instance take the cable to be the National Telephone Company's Standard Telephone Cable, which has the following constants per loop mile, that is per mile run of lead and return, $R = 88$ ohms, $C = \cdot05$ microfarad, $L = \cdot001$ henry, $S = 0$.

Suppose we apply a simple periodic $E.M.F.$ at one end of such a cable infinite or very great in length.

Let the frequency n be 83, which gives $p = 2\pi n = 500$ nearly. We have then

$$Lp=\frac{1}{2},\ Cp=\frac{25}{10^6},\ \sqrt{R^2+p^2L^2}=88.$$

Hence
$$2\,\alpha^2=\left(\frac{25}{10^6}\,88-\frac{12\cdot5}{10^6}\right)$$

$$2\,\beta^2=\left(\frac{25}{10^6}\,88+\frac{12\cdot5}{10^6}\right)$$

or
$$\alpha=\cdot034,\quad \beta=\cdot034,$$

and
$$\lambda=\frac{2\,\pi}{\beta}=185 \text{ miles}$$

$$W=\frac{p}{\beta}=15{,}000 \text{ miles per second.}$$

Next suppose $n = 830$ or $p = 5{,}000$.

Then $$Lp=5,\ Cp=\frac{1}{4000},\ \sqrt{R^2+p^2 L^2}=88{\cdot}1$$

$$2\ a^2=\left(\frac{88}{4000}-\frac{1}{800}\right),\ a=\cdot 104$$

$$2\ \beta^2=\left(\frac{88}{4000}+\frac{1}{800}\right),\ \beta=\cdot 108.$$

Hence $\lambda = 62{\cdot}8$ miles, $W = 50{,}000$ miles per second. Finally, if $n = 8{,}300$, and $p = 50{,}000$, we find that $a = \cdot 253$, $\beta = \cdot 435$, and $\lambda = 16$ miles, $W = 125{,}000$ miles per second.

This cable is therefore very far from being distorsionless.

As the frequency continually increases the wave velocity approximates to the velocity of light, viz., 186,000 miles per second, which, however, it can never exceed.

2. Propagation of Simple Periodic Currents along a Cable of Finite Length.—We have next to consider the modifications produced in the above formulæ when the cable is finite in length. This is the case which presents itself in practice. We then find that the reflection of the current or potential wave at the receiving and sending ends of the cable introduces considerable modifications into the above formulæ.

Returning to the general expressions for the potential and current at all points in an infinite cable, viz.,

$$V=A\ \epsilon^{-Px}+B\ \epsilon^{+Px} \quad . \quad . \quad . \quad . \quad (19)$$

$$I=\frac{P}{R+jpL}\left\{A\ \epsilon^{-Px}-B\ \epsilon^{+Px}\right\} \quad . \quad . \quad (20)$$

Let us write for ϵ^{-Px}, Cosh Px − Sinh Px, and for ϵ^{+Px}, Cosh Px + Sinh Px, and rearrange terms. We then transform the above equations into

$$V=(A+B)\ \text{Cosh}\ Px-(A-B)\ \text{Sinh}\ Px \quad . \quad . \quad (21)$$

$$I=\frac{1}{Z_0}\left\{(A-B)\ (\text{Cosh}\ Px-(A+B)\ \text{Sinh}\ Px\right\} \quad . \quad (22)$$

Now if $x = 0$, Sinh $Px = 0$, and Cosh $Px = 1$, and if we call V_1 and I_1 the potential difference and current at the sending end, then when $x = 0$, we find that

$$(A+B)=V_1 \text{ and } (A-B)=I_1\, Z_0$$

Suppose that the potential and current at the receiving end denoted by V_2 and I_2 and that the cable has a length l. Then at a distance x from the sending end and $l-x$ from the receiving end, if the potential and current are V and I, we can write the expressions for V and I in two forms, viz.,

$$V = V_1 \operatorname{Cosh} Px - I_1 Z_0 \operatorname{Sinh} Px \quad . \quad . \quad . \quad . \quad (23)$$

$$= V_2 \operatorname{Cosh} P(l-x) + I_2 Z_0 \operatorname{Sinh} P(l-x) \quad . \quad . \quad (24)$$

$$I = I_1 \operatorname{Cosh} Px - \frac{V_1}{Z_0} \operatorname{Sinh} Px \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$= I_2 \operatorname{Cosh} P(l-x) + \frac{V_2}{Z_0} \operatorname{Sinh} P(l-x) \quad . \quad . \quad (26)$$

The equations (23) and (25) are obtained from (21) and (22) by substituting V_1 for $A + B$ and $I_1 Z_0$ for $A - B$.

The equations (24) and (26) are obtained by reckoning the distance from the receiving end and assuming the voltage and current at that end to correspond to $x = 0$. The signs are changed because in the last case the current and voltage increase along the cable with distance reckoned from the receiving end.

The above equations (23) and (25) give the complete solution of the problem for the case of a finite cable, and we have three cases to consider, viz., (i.) when the receiving end is free or insulated, (ii.) when the receiving end is short circuited, and (iii.) when it is closed by a receiving instrument of known impedance.

3. Propagation of Currents along a Finite Cable Free or Insulated at the Receiving End.—In this case the current I_2 must be zero. Hence in the general equations corresponding to $x = l$ we must have $I = 0$, and making this substitution in equation (25) this gives us,

$$0 = I_1 \operatorname{Cosh} Pl - \frac{V_1}{Z_0} \operatorname{Sinh} Pl \quad . \quad . \quad . \quad (27)$$

or

$$I_1 Z_0 = V_1 \operatorname{Tanh} Pl \quad . \quad . \quad . \quad . \quad (28)$$

Substituting this value for $I_1 Z_0$ in (23) we have

$$V = V_1 [\operatorname{Cosh} Px - \operatorname{Tanh} Pl \operatorname{Sinh} Px] \quad . \quad . \quad (29)$$

This equation gives us the potential difference V (maximum value) at any distance x along a cable having a Propagation Constant P which is open at the far end.

Again from (28) we have

$$\frac{V_1}{I_1}=Z_1=Z_0 \text{ Coth } Pl \quad . \quad . \quad . \quad . \quad (30)$$

The ratio of the applied voltage to the current at the sending end is called the *final sending-end impedance* and denoted by Z_1. The reader should carefully distinguish between the *final sending-end impedance* $Z_1 = V_1/I_1$ and the *initial sending-end impedance* $Z_0 = \sqrt{R+jpL}/\sqrt{S+jpC}$.

If we compare the above expressions for V and V_1/I_1 for the finite cable with the corresponding expressions for the infinite cable the reader will at once see how the hyperbolic expressions are modified when there is reflection at the ends of the cable. For in the case of the infinite cable we have seen that

$$V=V_1\,\epsilon^{-Px}=V_1\,[\text{Cosh } Px-\text{Sinh } Px] \quad . \quad . \quad (31)$$

and
$$\frac{V_1}{I_1}=Z_0$$

whilst for the finite cable of length l we have

$$V=V_1\,[\text{Cosh } Px-\text{Tanh } Pl \text{ Sinh } Px] \quad . \quad . \quad (32)$$

and
$$\frac{V_1}{I_1}=Z_0 \text{ Coth } Pl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence the Tanh Pl and the Coth Pl sum up mathematically the effect of the reflections at the ends of the cable.

If in the equation (32) we put $x = l$ and therefore $V = V_2$ we have

$$V_2=V_1\,[\text{Cosh } Pl-\text{Tanh } Pl \text{ Sinh } Pl] \quad . \quad . \quad (34)$$

or
$$V_2=V_1 \text{ Sech } Pl \,. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This gives us an expression for the potential difference of the two sides of the cable at the far end when a voltage V_1 is applied at the sending end.

Again from (28) we have

$$I_1=\frac{V_1}{Z_0} \text{ Tanh } Pl \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and the two last expressions give us therefore the current into the cable at the sending end and the voltage at the distant end when that end is open.

Substituting this last value (36) for I_1 in the general equation (25) for the current, we have for the current I at any distance x the expression

$$I=\frac{V_1}{Z_0}[\text{Tanh } Pl \text{ Cosh } Px-\text{Sinh } Px] \quad . \quad . \quad (37)$$

If we refer back to equation (13) for the current in an infinite cable at any distance x from the sending end we see that it can be written

$$I=\frac{V_1}{Z_0}[\text{Cosh } Px-\text{Sinh } Px] \quad . \quad . \quad . \quad (38)$$

and on comparing the last two equations it will be evident that the effect of making the cable finite in length is to introduce the quantity Tanh Pl in both the formulæ for the current and voltage at any point. Thus for the infinitely long cable the equations

$$V=V_1[\text{Cosh } Px-\text{Sinh } Px] \quad . \quad . \quad . \quad (39)$$

and
$$I=\frac{V_1}{Z_0}[\text{Cosh } Px-\text{Sinh } Px] \quad . \quad . \quad . \quad (40)$$

give us the voltage and current at any distance x from the sending end, whilst for the finite cable of length l we have

$$V=V_1[\text{Cosh } Px-\text{Tanh } Pl \text{ Sinh } Px] \quad . \quad . \quad (41)$$

and
$$I=\frac{V_1}{Z_0}[\text{Tanh } Pl \text{ Cosh } Px-\text{Sinh } Px] \quad . \quad . \quad (42)$$

These formulæ show us that the values for the current and voltage in an infinite cable become greatly modified when we cut off a length and make it finite in length.

The reason for this is, as above stated, that when the cable is finite in length the current and voltage at any point are due to the superposition of an infinite number of effects due to the repeated reflection at the ends. We may in fact, as Dr. A. E. Kennelly has shown, derive the formulæ for the cable of finite length by a process of summation of these direct and reflected currents.[1]

Thus suppose a voltage V_1 is applied at the sending end, this travels up the cable of length l and at the far end becomes attenuated to $V_1\epsilon^{-Pl}$.

[1] See A. E. Kennelly, "On the Process of building up the Voltage and Current in a Long Alternating Current Circuit," *Proc.* of the American Academy of Arts and Sciences, Vol. XLII., p. 710, May, 1907.

At the open end this potential difference is reflected and doubled on reflection by the summation of the arriving and reflected potentials. Hence it jumps up on reflection to $2\, V_1 \epsilon^{-Pl}$. The reflected wave of potential runs back attenuating as it goes to $V_1\, \epsilon^{-2Pl}$, and is reflected at the closed sending end of the cable with sign changed as explained in Chapter II. The reflected wave again returns to the receiving end, at which it has attenuated to $-V_1\, \epsilon^{-3Pl}$ and this is doubled on reflection to $2\, V_1 \epsilon^{-3Pl}$ and so on.

Hence at the receiving end the actual potential difference is the sum of all these separate voltages, or

$$V_2 = 2\, V_1 (\epsilon^{-Pl} - \epsilon^{-3Pl} + \epsilon^{-5Pl} - \epsilon^{-7Pl} \text{ etc.}) \quad . \quad . \quad (43)$$

The series in the brackets is a geometrical progression with ratio $-\epsilon^{-2Pl}$ and hence we have,

$$V_2 = V_1 \frac{2\, \epsilon^{-Pl}}{1 + \epsilon^{-2Pl}} = V_1 \frac{2}{\epsilon^{Pl} + \epsilon^{-Pl}} = \frac{V_1}{\text{Cosh } Pl}$$

or

$$V_2 = V_1 \text{ Sech } Pl \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The hyperbolic function Sech Pl thus sums up the effect of all the repeated reflections at the ends.

The student will be assisted to comprehend the nature of this process by considering a similar effect in the case of light. Suppose a candle placed in an otherwise dark room. The illumination at any point would have a certain value depending on the distance from the candle. If then a mirror were placed at this point, the illumination just in front of the mirror would be equal to that due to the candle together with that due to another candle assumed to be placed as far behind the mirror as the first candle is in front of it, in other words at the position of the optical image of the first candle, the mirror being then supposed to be removed. Hence the single mirror produces on the illumination the effect of a second candle. In other words it doubles the illumination. Imagine then that a second mirror is placed behind the candle so that the candle stands between the two mirrors; the result will be that certain rays will be reflected backwards and forwards and the illumination at a point anywhere between the mirrors will be the same as if the mirrors were removed and an infinite number of candles were placed in positions coinciding with the optical images of the single candle formed by repeated reflections in the mirrors.

It will be noticed that in all these formulæ we are concerned with the hyperbolic functions of complex angles. Since the propagation constant P and therefore the propagation length Px or Pl are complex quantities, viz., $ax+j\beta x$ or $al+j\beta l$ the hyperbolic functions are themselves vectors, and we must obtain their values by the rules given in Chapter I.

Thus $$P=a+j\beta \text{ and } Pl=al+j\beta l$$
and
$$\text{Cosh } Pl=\text{Cosh }(al+j\beta l)$$
$$=\text{Cosh } al \text{ Cos } \beta l+j \text{ Sinh } al \text{ Sin } \beta l.$$

Since $\text{Sech } Pl=\frac{1}{\text{Cosh } Pl}$ we can obtain the value of Sech Pl by reciprocating Cosh Pl after its vector value has been thrown into the form $A/\underline{\theta}$.

For example, suppose $a=0{\cdot}1$, $\beta=0{\cdot}1$, $l=10$.

Then $$Pl=1+j\,1=1{\cdot}414\ /\underline{45^0}.$$
$$\text{Cosh } Pl=\text{Cosh }(1+j\,1)=\text{Cosh } 1 \text{ Cos } 1+j \text{ Sinh } 1 \text{ Sin } 1.$$

The 1 here in Cos 1 and Sin 1 means an angle of 1 radian or $180/\pi$ degrees $=57°\ 17'\ 45''$. Hence from the tables
$$\text{Cosh }(1+j\,1)=1\cdot5431\times0\cdot541+j\,1\cdot1752\times0\cdot841$$
$$=0{\cdot}835+j\,0{\cdot}988=1\cdot3\ /\underline{49°\ 45'}.$$

Hence $$\text{Sech }(1+j\,1)=0{\cdot}77\ /\overline{49°\ 45'}.$$

Again, if $$a=0{\cdot}1,\ \beta=0{\cdot}1, \text{ and } l=20$$
$$Pl=2+j\,2=2\cdot828\ /\underline{45°}$$
$$\text{Cosh }(2+j\,2)=\text{Cosh } 2 \text{ Cos } 2+j \text{ Sinh } 2 \text{ Sin } 2$$
$$=-3\cdot7622\times0\cdot416+j\,3\cdot6269\times0\cdot909$$
$$=-1\cdot565+j\,3\cdot297=3\cdot66\ /\underline{115°\ 24'}.$$

Hence $$\text{Sech }(2+j\,2)=0{\cdot}27\ /\overline{115°\ 24'}.$$

If $$a=0{\cdot}1,\ \beta=0{\cdot}3,\ l=5,\ Pl=1\cdot6\ /\underline{71°35'}.$$
$$\text{Cosh }(0\cdot5+j\,1\cdot5)=\text{Cosh } 0\cdot5 \text{ Cos } 1\cdot5+j \text{ Sinh } 0\cdot5 \text{ Sin } 1\cdot5$$
$$=1\cdot1276\times0\cdot071+j\,0\cdot521\times0\cdot997$$
$$=0\cdot080+j\,0\cdot520=0\cdot526\ /\underline{81°\ 15'}$$
and
$$\text{Sech }(0\cdot5+j\,1\cdot5)=1\cdot9\ /\overline{81°\ 15'}.$$

It can be easily proved in the same way that if
$Pl=0\cdot15+j\,1\cdot5=1\cdot5\ /\underline{84°\ 17'}$ Sech $Pl=6\cdot0\ /\overline{64°\ 23'}$ nearly.

It will be seen therefore that for various ratios of β/a and values of $l\sqrt{a^2+\beta^2}$ the value of the *size* of Sech Pl may be greater than unity.

Referring then to the formula (35) for the ratio of the voltage at the open receiving end of a cable to that at the sending end, viz.,

$$\frac{V_2}{V_1} = \text{Sech } Pl$$

it is clear that since Sech Pl can have a size greater than unity, the size of V_2 or the numerical value of the voltage at the receiving end can be greater than the numerical value of the voltage at the sending end.

Thus, referring to the calculations just given, if $a = 0{\cdot}1$ and $\beta = 0{\cdot}3$ and the length of the cable is five miles, then since Sech Pl in this case is $1{\cdot}9\ \underline{/81^\circ\ 15'}$, it follows that the voltage across the cable ends at the receiving end is 1·9 times the voltage applied at the sending end. In other words there is a considerable rise in voltage along the cable, instead of a fall, entirely due to the action of reflection at the ends of the cable.

It is of course obvious that there will in general be a considerable difference in phase between the voltages at the sending and receiving ends, whilst the actual numerical value of the voltage at the open receiving end may be less than, equal to, or greater than that at the sending end.

4. Propagation of Current along a Line Short Circuited at the Receiving End.—We have next to consider the case of a line short circuited at the receiving end, having a simple periodic electromotive force V_1 applied at the sending end. Then the voltage V_2 at the receiving end is zero. Hence in the general equations (23) and (25), viz.,

$$V = V_1 \text{ Cosh } Px - I_1 Z_0 \text{ Sinh } Px \quad . \quad . \quad . \quad (45)$$

$$I = I_1 \text{ Cosh } Px - \frac{V_1}{Z_0} \text{ Sinh } Px \quad . \quad . \quad . \quad (46)$$

Let us put $V_2 = 0$, $I = I_2$, $x = l$, and eliminate I_1, then we have

$$I_2 = \frac{V_1}{Z_0} \text{ Cosech } Pl \quad . \quad . \quad . \quad . \quad (47)$$

This gives us the current through the short circuit at the receiving end. Also from the equation (45), putting $V=0$ and $x=l$, we have

$$0=V_1 \text{ Cosh } Pl-I_1 Z_0 \text{ Sinh } Pl.$$

Hence $$I_1=\frac{V_1}{Z_0} \text{ Coth } Pl \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and from (48) and (47) we have

$$\frac{I_1}{I_2}=\text{Cosh } Pl \quad . \quad . \quad . \quad (49)$$

or $$\frac{I_2}{I_1}=\text{Sech } Pl \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This gives us the ratio of the current at the receiving end to that at the sending end, and it is clear that, since the size of Sech Pl can be greater than unity, the current at the receiving end can be larger than the current at the sending end.

It is easy to show, as in the case of the cable open at the far end, that this increase is due to the accumulated effects of reflection at the two ends.

The ratio $V_1/I_1=Z_1$ is called the *final sending end impedance*, and from equation (48) it is seen that

$$Z_1=Z_0 \text{ Tanh } Pl \quad . \quad . \quad . \quad . \quad (51)$$

Since $Z_0=\dfrac{\sqrt{R+jpL}}{\sqrt{S+jpC}}$ is a vector quantity and since Tanh Pl is a vector, it follows that Z_1 is a vector quantity, and this impedance is said to be measured in vector-ohms, meaning that the size is measured in ohms but that an angle giving direction is appended. The ratio $V_1/I_2=Z_2$ is called the *final receiving end impedance*, and from equation (47) we have

$$\frac{V_1}{I_2}=Z_2=Z_0 \text{ Sinh } Pl \quad . \quad . \quad . \quad . \quad (52)$$

We can measure experimentally for any line the values of V_1, I_1, and I_2, and hence determine Z_1 and Z_2. Suppose the ratio V_1/I_1 is measured with the far end of the line open or insulated. Let this value of Z_1 be denoted by Z_f. Then from equation (30) we have

$$Z_f=Z_0 \text{ Coth } Pl.$$

Suppose V_1/I_1 is then measured with the far end of the line short circuited, and call the ratio under these conditions Z_c, then from (48) we have

$$Z_c = Z_0 \text{ Tanh } Pl.$$

Hence $Z_f Z_c = Z_0^2$, or

$$Z_0 = \sqrt{Z_f Z_c} \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Hence the initial sending end impedance is the geometric mean of the final sending end impedances with the far end open and the far end closed. This measurement is the best means of finding the value of $\sqrt{R+jpL}/\sqrt{S+jpC}$ for any actual line.

5. Propagation of Simple Periodic Currents along a Transmission Line having a Receiving Instrument of known Impedance at the End.—This is the practical telephone problem to the consideration of which all that has previously been given is preliminary. We assume that we have a line of known primary constants R, L, C, S, and therefore known attenuation constant a and wave length constant β, and that a receiving instrument of known impedance Z_r is inserted across the line at the receiving end. Assuming we apply a simple periodic electromotive force V_1 at the sending end, the problem before us is to calculate the current and voltage at the receiving end, or at any distance.

If V_2 and I_2 are the potential difference and current at the receiving end, then the impedance of the receiving instrument Z_r is defined by the relation $V_2 = I_2 Z_r$. As V_2 and I_2 can be measured by suitable methods, we can always find Z_r.

Referring again to the fundamental equations (23) and (25) we have

$$V = V_1 \text{ Cosh } Px - I_1 Z_0 \text{ Sinh } Px \quad . \quad . \quad . \quad (54)$$

$$I = I_1 \text{ Cosh } Px - \frac{V_1}{Z_0} \text{ Sinh } Px \quad . \quad . \quad . \quad (55)$$

Substituting for V, I, and x the values at the receiving end, viz., V_2, I_2, and l, we have

$$V_2 = I_2 Z_r = V_1 \text{ Cosh } Pl - I_1 Z_0 \text{ Sinh } Pl \quad . \quad . \quad (56)$$

$$I_2 = I_1 \text{ Cosh } Pl - \frac{V_1}{Z_0} \text{ Sinh } Pl \quad . \quad . \quad . \quad . \quad (57)$$

Eliminating I_1 we have

$$I_2=\frac{V_1}{Z_0 \text{ Sinh } Pl+Z_r \text{ Cosh } Pl} \quad . \quad . \quad . \quad (58)$$

and eliminating I_2 we have

$$I_1=\frac{V_1}{Z_0} \cdot \frac{Z_0 \text{ Cosh } Pl+Z_r \text{ Sinh } Pl}{Z_r \text{ Cosh } Pl+Z_0 \text{ Sinh } Pl} \quad . \quad . \quad . \quad (59)$$

These expressions give us the current at the receiving and sending ends respectively. Hence also

$$\frac{I_1}{I_2}=\text{Cosh } Pl+\frac{Z_r}{Z_0}\text{ Sinh } Pl \quad . \quad . \quad . \quad (60)$$

On comparing the above formula with the corresponding formula (49) for the cable short circuited at the receiving end, we see that the effect of the receiving instrument is to add a term $\frac{Z_r}{Z_0}$ Sinh Pl, and so make the ratio I_1/I_2 larger. It is possible, however, for I_2 to be greater than I_1. From the above formulæ (59) and (58) we can obtain expressions for the *final sending end impedance* $Z_1 = V_1/I_1$ and for the final receiving end impedance $Z_2 = V_1/I_2$, viz.,

$$Z_1=\frac{V_1}{I_1}=Z_0\frac{Z_r \text{ Cosh } Pl+Z_0 \text{ Sinh } Pl}{Z_0 \text{ Cosh } Pl+Z_r \text{ Sinh } Pl} \quad . \quad . \quad (61)$$

$$Z_2=\frac{V_1}{I_2}=Z_0 \text{ Sinh } Pl+Z_r \text{ Cosh } Pl \quad . \quad . \quad . \quad (62)$$

The above expressions can be simplified by taking advantage of two well-known theorems in circular and hyperbolic trigonometry.

Theorem I. If θ is any circular angle such that $\tan \theta = \frac{B}{A}$, and if ϕ is any other angle, then

$$A \text{ Sin } \phi+B \text{ Cos } \phi=\sqrt{A^2+B^2}\text{ Sin }(\phi+\theta).$$

If $B/A = \tan \theta$, then

$$\frac{B}{\sqrt{A^2+B^2}}=\text{Sin } \theta \text{ and } \frac{A}{\sqrt{A^2+B^2}}=\text{Cos } \theta,$$

but Sin $(\phi + \theta) =$ Sin ϕ Cos θ + Cos ϕ Sin θ. Hence, substituting the values of Sin θ and Cos θ, we have

$$A \text{ Sin } \phi+B \text{ Cos } \phi=\sqrt{A^2+B^2}\text{ Sin }(\phi+\theta) \quad . \quad . \quad (63)$$

Theorem II. If γ is any hyperbolic angle such that $\tanh \gamma = \frac{B}{A}$, and if δ is any other hyperbolic angle, then

$$A \text{ Sinh } \delta + B \text{ Cosh } \delta = \sqrt{A^2 - B^2} \text{ Sinh } (\delta + \gamma).$$

For
$$\text{Tanh } \gamma = \frac{\text{Sinh } \gamma}{\text{Cosh } \gamma} = \frac{B}{A}$$
and
$$\text{Cosh }^2\gamma - \text{Sinh }^2\gamma = 1.$$

Hence
$$\text{Sinh } \gamma = \frac{B}{\sqrt{A^2 - B^2}} \text{ and Cosh } \gamma = \frac{A}{\sqrt{A^2 - B^2}}.$$

But
$$\text{Sinh } (\delta + \gamma) = \text{Sinh } \delta \text{ Cosh } \gamma + \text{Cosh } \delta \text{ Sinh } \gamma$$

Hence
$$A \text{ Sinh } \delta + B \text{ Cosh } \delta = \sqrt{A^2 - B^2} \text{ Sinh } (\delta + \gamma) \quad . \quad . \quad (64)$$

Again, from the fundamental equation (23)

$$V_2 = V_1 \text{ Cosh } Pl - I_1 Z_0 \text{ Sinh } Pl \quad . \quad . \quad . \quad (65)$$

and from the value obtained for I_1 in (59) we have

$$I_1 Z_0 = V_1 \frac{Z_0 \text{ Cosh } Pl + Z_r \text{ Sinh } Pl}{Z_r \text{ Cosh } Pl + Z_0 \text{ Sinh } Pl} \quad . \quad . \quad (66)$$

Hence, substituting (66) in (65), we have

$$V_2 = V_1 \left\{ \text{Cosh } Pl - \left(\frac{Z_0 \text{ Cosh } Pl + Z_r \text{ Sinh } Pl}{Z_r \text{ Cosh } Pl + Z_0 \text{ Sinh } Pl} \right) \text{Sinh } Pl. \right\} \quad (67)$$

or since $\text{Cosh }^2Pl - \text{Sinh }^2Pl = 1$ we have

$$V_2 = \frac{V_1 Z_r}{Z_0 \text{ Sinh } Pl + Z_r \text{ Cosh } Pl} \quad . \quad . \quad . \quad (68)$$

Accordingly by the aid of the *Theorem II.* we can write the formulæ for the currents and final impedances as follows :—

$$I_2 = \frac{V_1}{\sqrt{Z_0^2 - Z_r^2}} \text{ Cosech } (Pl + \gamma) \quad . \quad . \quad . \quad (69)$$

$$I_1 = \frac{V_1}{Z_0} \text{ Coth } (Pl + \gamma) \quad . \quad . \quad . \quad . \quad (70)$$

$$Z_2 = \sqrt{Z_0^2 - Z_r^2} \text{ Sinh } (Pl + \gamma) \quad . \quad . \quad . \quad (71)$$

$$Z_1 = Z_0 \text{ Tanh } (Pl + \gamma) \quad . \quad . \quad . \quad . \quad (72)$$

where
$$\text{Tanh } \gamma = \frac{Z_r}{Z_0} \text{ or } \gamma = \text{Tanh}^{-1} \left(\frac{Z_r}{Z_0} \right) \quad . \quad . \quad . \quad (73)$$

Hence it follows that

$$I_2 = I_1 \text{ Cosh } \gamma \text{ Sech } (Pl + \gamma) \quad . \quad . \quad . \quad (74)$$

Also from (68), bearing in mind that $\text{Tanh } \gamma = \frac{Z_r}{Z_0}$ and therefore $\text{Sinh } \gamma = \frac{Z_r}{\sqrt{Z_0^2 - Z_r^2}}$, we can express the ratio V_2/V_1 by

$$V_2 = V_1 \text{ Sinh } \gamma \text{ Cosech } (Pl + \gamma) \quad . \quad . \quad . \quad (75)$$

A consideration of these last five formulæ and comparison of them with the similar formulæ for the short circuited cable shows that the introduction of the receiving instrument of impedance Z_r has the same effect as if the line were made longer by an amount l' such that $Pl' = \gamma$ and was then short circuited at the receiving end. At the same time the effect of this lengthening is to cause an alteration in the effective initial sending end impedance as far as the current at the receiving end is concerned, but not for the sending end current.

We have shown (equation (52)) that the final receiving end impedance V_1/I_2 in the case of a line short circuited at the receiving end is $Z_2 = Z_0$ Sinh Pl.

And also that the same quantity for the line with receiving instrument of impedance Z_r at the end is (by equation (62)) given by

$$Z_2 = Z_0 \text{ Sinh } Pl + Z_r \text{ Cosh } Pl.$$

Hence if we denote the final receiving end impedance of the short circuited line by Z^1_2 we have

$$\frac{Z_2}{Z^1_2} = 1 + \frac{Z_r}{Z_0} \text{ Coth } Pl \quad . \quad . \quad . \quad . \quad (76)$$

When the line is very long Coth Pl approximates to unity and then

$$\frac{Z_2}{Z^1_2} = 1 + \frac{Z_r}{Z_0} = \frac{Z_0 + Z_r}{Z_0} \quad . \quad . \quad . \quad . \quad (77)$$

CHAPTER IV

TELEPHONY AND TELEPHONIC CABLES

1. The Principles of Telephony.—Telephony is the art and science of transmitting articulate speech by means of electric currents between two places connected by a wire or cable. The conductor may be either a pair of overhead wires or a single wire with earth return, or a twin cable.

At one end of this conductor is placed a *telephone transmitter*, which comprises, generally speaking, an induction coil, the secondary circuit of which is connected to the pair of line wires or to the line wire and the earth. In the primary circuit of the coil is included a battery and a microphone. This last consists in one form of a shallow circular metal box with a solid back; closed in front by a diaphragm of flexible metal which is insulated by a ring of ebonite from the box itself.

The cavity is filled with granulated graphitic carbon. Wires are connected to the diaphragm and to the box.

An electric circuit is thus formed, of which the granulated carbon is part.

This arrangement constitutes the microphone, and it is joined in series with the battery and with the primary circuit of the induction coil. If the carbon granules are compressed by pressing in the diaphragm the resistance of the circuit is reduced and more current flows through the primary circuit of the coil and hence induces a current in the secondary circuit, which flows through the line.

If articulate speech is made in front of the diaphragm the rapid changes of air pressure which constitute sound cause a corresponding movement of the diaphragm and therefore equivalent changes in resistance in the carbon granules. Hence a secondary current is sent into the line the variations in which more or less perfectly follow the changes of air pressure in front of the diaphragm.

The motion of the air molecules when transmitting a sound wave is to and fro in the direction of transmission, but the amplitude of their acoustic motion is extremely small.

Lord Rayleigh determined the amplitude of this air motion for the sound of a whistle giving a note having a frequency of 2730, which was loud enough to be heard at a distance of 820 metres in every direction.[1] This amplitude he found to be 0·081 of one millionth of a centimetre or 0·00081 μ where μ is the thousandth part of a millimetre. This is about one thousandth part of the wave length of a ray of red light and shows how extremely small an air motion the normal human ear is capable of appreciating. In the case of articulate sounds this motion of the air particles is a highly irregular one, but in the case of musical sounds or prolonged vowel sounds the motion is a regularly repeated or cyclical one which is to and fro in the line of propagation of the sound. We can graphically represent it by the displacement of a point which moves uniformly along a straight line and at the same time executes a vibratory motion at right angles to that line which copies the to and fro motion of the air particle in the line of propagation. We then obtain for continuous sounds a wavy line which is called the *graph* or *wave form* of the sound.

The curves in Fig. 1 represent the wave forms of five vowel sounds, A, E, I, O, U, pronounced in the Continental manner. If the sound recorded is that of a tuning fork or open organ pipe gently blown the wave form is a simple periodic curve such that the displacement or ordinate y at any time t is given by the expression $y = Y \operatorname{Sin} pt$ where Y is the maximum ordinate and $p = 2\pi$ times the frequency n.

On the other hand, if the sound is a consonantal sound or noise, the wave form is an irregular non-repeated curve. If it is a periodic or repeated curve the maximum amplitude is determined by the loudness of the sound and its wave length or period by the pitch.

[1] See Lord Rayleigh, *Proc. Roy. Soc.*, Vol. XXVI., p. 248, 1877, or *Collected Papers*, Vol. I., p. 328.

FIG. 1.—Wave Form of the vowel sounds A, E, I, O, U, taken with the Oscillograph.

When any sound or speech is made in proximity to the diaphragm of the microphone the aerial vibrations create a more or less similar motion of the diaphragm, and a variation in the resistance of the carbon granules takes place, which in turn causes the current into the line wire to be varied in a manner somewhat similar to the variations of the air pressure and motion in front of the diaphragm.

Owing to the fact that the diaphragm has a natural period of vibration of its own, this current variation in the line is not an exact copy of the air pressure variation, but it is sufficiently like it to achieve practical telephony. We may then assume that there is in the line wire a current the wave form of which at the sending end is somewhat similar to that of the wave form of the air motion.

This current flows along the line, and being a periodic current it is attenuated as it flows. At the receiving end it enters the telephone receiver, which is generally a Bell magneto telephone consisting of a permanent magnet of bar or horse-shoe form, round the poles of which are coils of wire inserted in series with the line wire. In close proximity to the poles of this magnet is a flexible diaphragm of iron (ferrotype plate). When the periodic current from the line flows through the coils wound on the magnet it slightly increases or decreases the magnetism and attracts more or less the iron diaphragm. The result is that the diaphragm of the receiving instrument experiences vibrations which are approximately a copy of the variations of the line current and therefore of the vibrations of the diaphragm of the transmitter. The air in proximity to the receiving diaphragm is therefore set in vibration in a manner which is not very dissimilar to that of the diaphragm of the transmitter and therefore to that of the air in proximity to the latter. We thus repeat in a distant place sounds made near the transmitter. This repetition is, however, far from being perfect. The transmission of articulate sounds is wonderfully assisted by the power of the human intelligence to guess from a very imperfect repetition the significance of the sound as a word spoken to the transmitting diaphragm. There is, however, a limit to this guesswork, and beyond a certain point a sound may be heard but it has no meaning. This constitutes the limitation of telephony, and we have in the next place to consider the causes of this limitation.

2. Fourier's Theorem.—If we have any single valued periodic curve, that is one having only one value of the ordinate to one value of the abscissa and repeating itself at regular intervals, then, no matter how irregular the curve may be provided it does not exhibit discontinuities, it is always possible to imitate this curve exactly by adding the ordinates of superimposed simple periodic or sine curves of suitable amplitude and phase difference, having wave lengths which are in integer relation to each other. Thus, for instance, if we draw sine curves having wave lengths in the ratio of 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., we can cut thin sheets of zinc so that these curves form the outline of one edge (see Fig. 2), and these templates can be used to draw sine curves of certain relative amplitude and phase difference relatively to one another.

We can then add together the ordinates of the several components corresponding to any one abscissa, to form the ordinate of a new compound curve which is then said to be made by the synthesis of these several sine curves. There is no difficulty in carrying out this synthetic process.

It is rather more difficult to perform the inverse process, *i.e.*, when given an irregular but periodic single valued curve to find the sine components of which it is built up, but it can be done in virtue of Fourier's Theorem, which is as follows:—

Let y be the ordinate of any periodic curve which is single valued and without discontinuities; then y can be expressed by the series

$$y=A_0+A_1 \operatorname{Sin} pt+B_1 \operatorname{Cos} pt+A_2 \operatorname{Sin} 2pt+B_2 \operatorname{Cos} 2pt$$
$$+A_3 \operatorname{Sin} 3pt+B_3 \operatorname{Cos} 3pt \text{ etc.} \quad . \quad . \quad . \quad (1)$$

where $p=2\pi/T$ and T is the periodic time of the fundamental sine curve assuming it to be described by a point which moves with uniform velocity in a horizontal direction.

Accordingly pt is the abscissa corresponding to y, such abscissa being measured from the zero point of the fundamental sine curve.

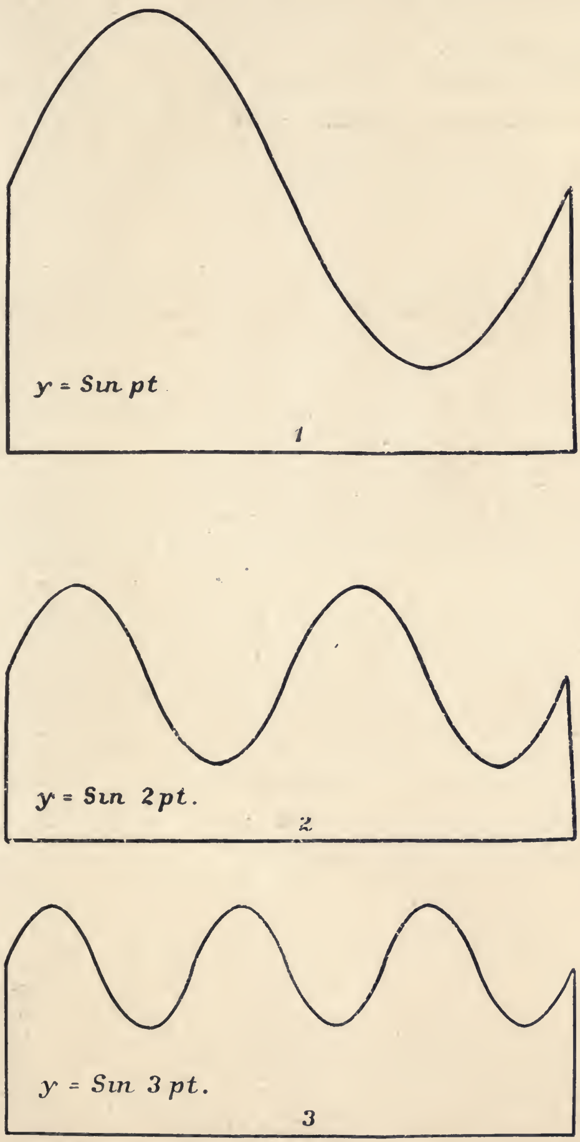

FIG. 2.—Templates of Curves representing Harmonic Sine Curves.

The problem then is to find the value of the constants A_0, A_1, B_1, A_2, B_2, etc. This can be done by the aid of the following theorems:—

1. If θ is any angle, then the average value of Sin θ or Cos θ taken at equal small angular distances over four right angles is zero.

2. The average value of both $\text{Sin}^2\,\theta$ and $\text{Cos}^2\,\theta$ taken in the same way is $\frac{1}{2}$.

3. The average value of Sin θ Cos θ and of Sin $n\,\theta$ Sin $m\,\theta$ or Cos $n\,\theta$ Cos $m\,\theta$ taken in the same manner over four right angles is zero.

The truth of *Theorem* 1 is obvious. If we describe a sine curve extending over a complete period, taking this as 360°, and draw equi-spaced ordinates, then it is clear that for every positive ordinate there will be an equal negative ordinate separated by abscissæ equal to 180°, and when we add their values algebraically the sum of the whole number is zero, and therefore the average value is zero.

Theorem 1 expressed in the language of the integral calculus is otherwise proved as follows:—

$$M\,\text{Sin}\,\theta = \frac{1}{2\pi}\int_0^{2\pi} \text{Sin}\,\theta\,d\,\theta \quad . \quad . \quad . \quad . \quad (2)$$

where M Sin θ stands for the average or mean value of Sin θ taken at equal angular intervals $d\theta$ between 0 and 2π. If then $\theta = pt = \frac{2\pi}{T}t$, then

$$M\,\text{Sin}\,pt = \frac{1}{T}\int_0^{T} \text{Sin}\,pt\,dt \quad . \quad . \quad . \quad . \quad (3)$$

$$= \frac{1}{T}\,{}_0^T\!\Big[-\text{Cos}\,pt\Big]$$

$$= \frac{1}{T}\Big(-1+1\Big) = 0.$$

Theorem 2 can be proved as follows:—

Since $\quad \text{Cos}\,2\theta = \text{Cos}^2\theta - \text{Sin}^2\theta = 1 - 2\,\text{Sin}^2\theta = 2\,\text{Cos}^2\theta - 1$

we have $$\text{Sin}^2\,\theta = \frac{1}{2} - \frac{1}{2}\,\text{Cos}\,2\theta$$

and $$\text{Cos}^2\,\theta = \frac{1}{2} + \frac{1}{2}\,\text{Cos}\,2\theta.$$

The average value of $\text{Sin}^2\theta$ must therefore be $\frac{1}{2}$ because the average value of $\text{Cos}\ 2\theta$ taken at small equal angular intervals between $\theta = 0$ and $\theta = 2\pi$ is zero. The same for $\text{Cos}^2\theta$.

Theorem 3 is also easily proved; for

$$\text{Sin}\ (n+m)\ \theta = \text{Sin}\ n\theta\ \text{Cos}\ m\theta + \text{Cos}\ n\theta\ \text{Sin}\ m\theta$$
$$\text{Sin}\ (n-m)\ \theta = \text{Sin}\ n\theta\ \text{Cos}\ m\theta - \text{Cos}\ n\theta\ \text{Sin}\ m\theta.$$

Hence $\text{Sin}\ n\theta\ \text{Cos}\ m\theta = \frac{1}{2}\ \text{Sin}\ (n+m)\ \theta + \frac{1}{2}\ \text{Sin}\ (n-m)\ \theta.$

Accordingly whatever n and m may be, the average value of $\text{Sin}\ n\ \theta\ \text{Cos}\ m\ \theta$ must be zero because the average values of $\text{Sin}\ (n+m)\ \theta$ and $\text{Sin}\ (n-m)\ \theta$ are individually zero.

Again it can be proved in the same way that

$$\text{Sin}\ n\theta\ \text{Sin}\ m\theta = \frac{1}{2}\ \text{Cos}\ (n-m)\theta - \frac{1}{2}\ \text{Cos}\ (n+m)\theta$$

$$\text{Cos}\ n\theta\ \text{Cos}\ m\theta = \frac{1}{2}\ \text{Cos}\ (n-m)\theta + \frac{1}{2}\ \text{Cos}\ (n+m)\theta.$$

It is clear then that the average values of $\text{Sin}\ n\ \theta\ \text{Sin}\ m\ \theta$ and $\text{Cos}\ n\ \theta\ \text{Cos}\ m\ \theta$ are zero except when $n = m$, in which case their average values are $\frac{1}{2}$.

Returning then to the expression first given by Fourier for the ordinate y of a single valued continuous curve by the series

$$y = A_0 + A_1\ \text{Sin}\ pt + B_1\ \text{Cos}\ pt + A_2\ \text{Sin}\ 2\ pt + B_2\ \text{Cos}\ 2\ pt \text{ etc.},$$

we have to show how the constants in this series can be found.

Suppose a number of equi-spaced ordinates y_1, y_2, y_3 to be drawn to the curve over one complete period. Then the average value of these ordinates throughout this period is the value of A_0 because the average value of all the Sine and Cosine terms is zero.

Again let us multiply both sides by $\text{Sin}\ pt$ and take the average value throughout the period, we have

$$y\ \text{Sin}\ pt = A_0\ \text{Sin}\ pt + A_1\ \text{Sin}^2\ pt + B_1\ \text{Cos}\ pt\ \text{Sin}\ pt, + \text{ etc.}$$

When we take the average, the value of all the terms on the right-hand side is zero, except $A_1\ \text{Sin}^2 pt$ which is equal to $\frac{A_1}{2}$.

Hence we have the average value of $y\ \text{Sin}\ pt = \frac{A_1}{2}$ or

$A_1 =$ twice the average value of $y\ \text{Sin}\ pt$ through a period.

In the same way by multiplying successively by Cos pt, Sin 2 pt, Cos 2 pt, etc., we can prove that

B_1 = twice the average value of y Cos pt,

A_2 = twice the average value of y Sin 2 pt,

B_2 = twice the average value of y Cos 2 pt, etc.

Accordingly we have the following rule for analysing a compound periodic curve into its constituent harmonics.

Rule up, say, 24 ordinates at equal distances throughout one complete period of the curve and measure off their lengths. Call them y_1, y_2, y_3, y_4, etc.

Then, since 360/24 = 15 we must look out in the Tables the values of Sin 0°, Sin 15°, Sin 30°, Sin 45°, Sin 60°, etc., and make a table in columns as follows :—

Column I. has the 24 numerical values, $y_1, y_2 \ldots \ldots y_{24}$ written down one above the other.

Column II. has the values y_1 Sin 15, y_2 Sin 30, y_3 Sin 45, etc., written above one another.

Column III. has the values of y_1 Cos 15, y_2 Cos 30, y_3 Cos 45, etc., written one above the other.

Column IV. has the values y_1 Sin 30, y_2 Sin 60, y_3 Sin 90, etc., written one above the other.

Column V. has the values y_1 Cos 30, y_2 Cos 60, y_3 Cos 90, etc., written one above the other.

Column VI. has the values y_1 Sin 45, y_2 Sin 90, y_3 Sin 135, and so on, regard being taken to the algebraic sign of the Sine or Cosine.

We have already shown (see Chap. III., § 5) that

$$A \text{ Sin } \phi + B \text{ Cos } \phi = \sqrt{A^2+B^2} \text{ Sin } (\phi+\theta) \quad . \quad . \quad (4)$$

where $\tan\theta = \frac{B}{A}$; hence we can write Fourier's theorem in the form,

$$y = A_0 + \sqrt{A_1^2+B_1^2} \text{ Sin } (pt+\theta_1) + \sqrt{A_2^2+B_2^2} \text{ Sin } (2\,pt+\theta_2) \text{ etc.} \quad . \quad (5)$$

In this case the quantities $\sqrt{A_1^2+B_1^2}$, $\sqrt{A_2^2+B_2^2}$, etc., are called the *amplitudes* of the different harmonics, and the angles θ_1, θ_2, etc., are called the *phase angles*.

If the curve is a periodic curve of such kind that for every ordinate of a certain length there is another ordinate half a wave length further on of equal length but opposite sign, then the first or constant term A_0 is zero, because the average value of all the equi-spaced ordinates is then zero.

As an example of the Fourier analysis of a complex periodic curve we may take the following [1] :—

The firm line curve in Fig. 3 is a curve formed by adding together the ordinates of three simple periodic or (dotted) sine

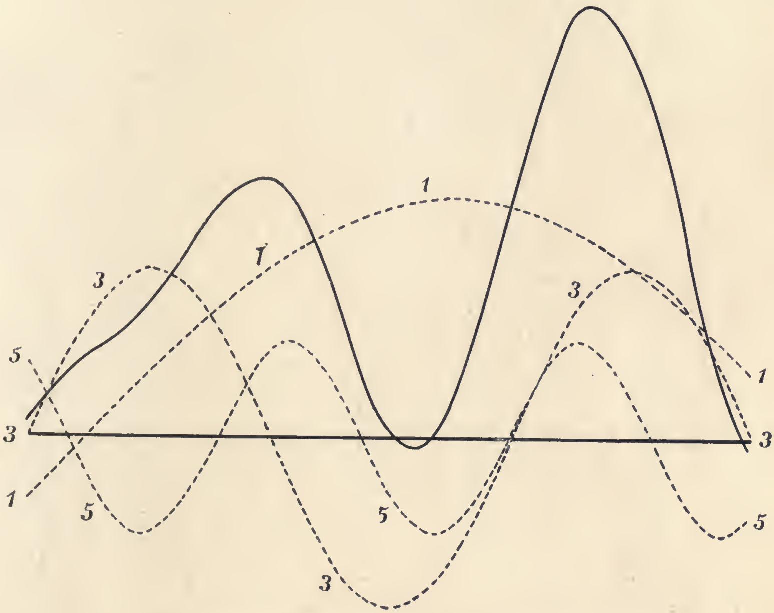

FIG. 3.—Fourier Analysis of a Periodic Curve.

curves of which the wave lengths are in the ratio of $1 : \frac{1}{3} : \frac{1}{5}$ and of which the amplitudes are respectively 4, 2·8, and 1·6. These curves are shifted relatively to one another so that the second harmonic lies 15° behind the first and the third about 4° 30′ behind the first harmonic. These harmonics are represented by the three dotted line curves in Fig. 3.

Hence the equation to the firm line curve is

$$y = 4 \text{ Sin } \phi + 2 \cdot 8 \text{ Sin } 3\,(\phi + 15^\circ) - 1 \cdot 6 \text{ Sin } 5\,(\phi + 4^\circ\, 30') . \quad (6)$$

[1] The method of numerical calculation here given was originally described by Professor J. Perry in *The Electrician*, Vol. XXVIII., p. 362, 1892.

If we shift the origin to the zero point of the principal sine curve, this is equivalent to substituting $pt - 15°$ for ϕ in the above equation, and the expression then becomes,

$$y = 4 \text{ Sin } (pt - 15°) + 2 \cdot 8 \text{ Sin } 3\, pt - 1 \cdot 6 \text{ Sin } (5\, pt - 52° \, 30') \quad . \quad (7)$$

We then take the curve as drawn and rule up 12 equi-spaced ordinates at intervals of 15° and find by actual measurement that these ordinates have the values 0, 1·5, 2·4, 3·8, 4·0, 2·3, −0·1, 0·4, 4·2, 7·0, 6·2, 2·7 and 0.

We then proceed to make two tables as follows:—Table I. contains the values of Sin pt, Cos pt, Sin $3pt$, Cos $3pt$, Sin $5pt$, Cos $5pt$ for values of pt from 0° to 180°.

TABLE I.

pt.	Sin pt.	Cos pt.	$3\,pt$.	Sin $3\,pt$.	Cos $3\,pt$.	$5\,pt$.	Sin $5\,pt$.	Cos $5\,pt$.
0	0	1·000	0	0	1·000	0	0	1·000
15	·259	·966	45	·707	·707	75	·966	·259
30	·500	·866	90	1·000	0	150	·500	−·866
45	·707	·707	135	·707	−·707	225	−·707	−·707
60	·866	·500	180	0	−1·000	300	−·866	·500
75	·966	·259	225	−·707	−·707	15	·259	·966
90	1·000	0	270	−1·000	0	90	1·000	0
105	·966	−·259	315	−·707	·707	165	·259	−·966
120	·866	−·500	360	0	1·000	240	−·866	−·500
135	·707	−·707	45	·707	·707	315	−·707	·707
150	·500	−·866	90	1·000	0	30	·500	·866
165	·259	−·966	135	·707	−·707	105	·966	−·259
180	0	−1·000	180	0	−1·000	180	0	−1·000

In Table II., Column II., are tabulated the measured values of the 12 ordinates y of the firm line curve taken at equi-spaced distances over the half wave length represented by 180°. In Columns III. to VIII. are tabulated the values of y Sin pt, y Cos pt, y Sin $3\,pt$, y Cos $3\,pt$, y Sin $5\,pt$, y Cos $5\,pt$, and at the foot of each column is given the mean value of each series of numbers; also twice the mean values, which are as shown above, are the values of the Constants A_1, B_1, A_3, B_3, A_5, B_5 respectively. From this are calculated the values of the amplitudes $\sqrt{A_1^2 + B_1^2}$, $\sqrt{A_3^2 + B_3^2}$, $\sqrt{A_5^2 + B_5^2}$, and the phase angle tangents B_1/A_1, B_3/A_3, and B_5/A_5.

Hence we can find the phase angles themselves and arrive at an expression for the ordinate of the dotted curve expressed as a Fourier series. On comparing the expression thus obtained by calculation, viz.,

$$y = 3 \cdot 92 \text{ Sin } (pt - 15° \ 50') + 2 \cdot 9 \text{ Sin } (3\,p + 0° \ 50') \\ - 1 \cdot 55 \text{ Sin } (5\,pt - 51° \ 30') \quad . \quad . \quad . \quad (8)$$

with the expression from which the curve was drawn as given under Fig. 3, viz.,

$$y = 4{\cdot}0 \text{ Sin } (pt - 15°) + 2 \cdot 8 \text{ Sin } 3\,pt - 1 \cdot 6 \text{ Sin } (5\,pt - 52° \ 30') \ . \quad (9)$$

it will be seen that there are small differences in the amplitudes and phase angles, but that the calculated value of the expression agrees substantially with the expression from which the firm line curve in Fig. 3 was drawn. The differences, such as they are, are due to the fact that we have only measured 12 ordinates in the half wave, but it would require a larger number to secure a better agreement.

TABLE II.

I. *pt.*	II. *y.*	III. *y* Sin *pt.*	IV. *y* Cos *pt.*	V. *y* Sin 3 *pt.*	VI. *y* Cos 3 *pt.*	VII. *y* Sin 5 *pt.*	VIII. *y* Cos 5 *pt.*
0°	0	0	0	0	0	0	0
15°	1·5	0·388	1·449	1·060	1·060	1·449	0·388
30°	2·4	1·200	1·838	2·400	0·000	1·200	2·078
45°	3·8	2·686	2·686	2·687	−2·687	−2·687	−2·687
60°	4·0	3·464	2·000	0·000	−4·000	−3·464	2·000
75°	2·3	2·222	0·586	−1·626	−1·626	0·596	2·222
90°	−0·1	−0·100	0·000	0·100	0·000	−0·100	0·000
105°	0·4	0·386	−0·104	−0·283	0·283	0·104	−0·386
120°	4·2	3·637	−2·100	0·000	4·200	−3·637	−2·100
135°	7·0	4·949	−4·949	4·949	4·949	−4·949	4·949
150°	6·2	3·100	−5·369	6·200	0·000	3·100	5·369
165°	2·7	0·699	−2·608	1·908	−1·908	2·608	−0·706
180°	0	0·000	0·000	0·000	0·000	0·000	0·000
Totals .		+22·731 −0·100	+8·559 −15·030	+19·304 −1·909	+10·492 −10·221	+9·057 −14·837	+15·228 −7·957
Net totals .		+22·631	−6·471	+17·395	+0·271	−5·780	+7·271
Mean values .		+1·886	−0·539	+1·450	+0·022	−0·482	+0·606
Twice mean values		+3·77 $=A_1$	−1·08 $=B_1$	+2·9 $=A_3$	+0·044 $=B_3$	−0·964 $=A_5$	+1·212 $=B_5$

Therefore $\sqrt{A_1^2 + B_1^2} = 3{\cdot}92$ $\quad \sqrt{A_3^2 + B_3^2} = 2{\cdot}9$ $\quad \sqrt{A_5^2 + B_5^2} = 1{\cdot}55$

And $\tan^{-1}\dfrac{B_1}{A_1} = \tan^{-1}(-{\cdot}283)$ $\quad \tan^{-1}\dfrac{B_3}{A_3} = \tan^{-1}({\cdot}015)$ $\quad \tan^{-1}\dfrac{B_5}{A_5} = \tan^{-1}(-1{\cdot}257)$

Hence we have $= \theta_1 = -15° \ 50'$ $\quad = \theta_3 = 0° \ 50'$ $\quad = \theta_5 = -51° \ 30'$

and $\quad y = 3{\cdot}92 \text{ Sin } (pt - 15° \ 50') + 2{\cdot}9 \text{ Sin } (3\,pt + 0° \ 50') - 1{\cdot}55 \text{ Sin } (5\,pt - 51° \ 30')$.

3. The Analysis and Synthesis of Sounds.—The analysis of a periodic curve into its constituent sine curves in accordance with Fourier's theorem is not merely a mathematical conception or process, but it is in accordance with the facts of acoustics.

We can by certain appliances cause the oscillatory motions of sounding bodies to record the nature of their vibrations in graphical form. Thus if we attach to the prong of a steel tuning fork a bristle and hold the vibrating fork near a rapidly revolving drum covered with smoked paper we can make the bristle record the wave form of the vibration upon the paper. It is found that this record is a sine curve. The aerial vibrations produced by the fork and also those produced by open organ pipes gently blown are in like manner simple sine vibrations. Such sounds are smooth and not unpleasant to the ear, but they are wanting in character or brilliancy. If, however, a special sound such as a continuous vowel sound is made, we find by experiments with the oscillograph or phonograph that the wave form is very irregular although periodic. Von Helmholtz was led by these considerations to his classical experiment of the synthesis of vowel sounds. He provided a number of tuning forks the frequencies of which were in the ratio $1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4}$, etc., and each tuning fork had a hollow brass sphere in proximity to it, the said sphere having an opening in it. These spheres are called resonators, and when constructed of such size that the corresponding tuning fork can set the air in it in vibration they re-enforce the sound, provided the aperture of the resonator is open. The tuning forks were maintained in vibration continuously by electromagnets, and by means of keys the operator could more or less open the aperture of any resonator and so mix together sounds of harmonic frequencies in various proportions as regards amplitude or loudness. Von Helmholtz found that he was thus able to imitate various vowel sounds, and that these latter are therefore compounded of various simple sine vibrations of different amplitude. The question then arises, has the relative difference of phase of the simple sine components anything to do with the production of the quality of the sound?

We know from Fourier's theorem that the wave form of the complex curve depends not only on the amplitudes but on the relative phase of the component sine curves. The question then arises whether the ear when impressed by a complex vibration takes note of the difference of phase as well as the difference in amplitude of the component harmonics.

Von Helmholtz drew the conclusion from his experiments that the quality of the sound depended only on the amplitudes of the harmonics and not on their relative phase (see Helmholtz's book "Sensations of Tone," English translation by Ellis, Chap. VI., p. 126).

Helmholtz's conclusion is not generally accepted. Lord Rayleigh (see "Theory of Sound," Vol. II., Chap. XXIII.) has given arguments to prove that the difference of phase is not without effect. Also König, another great acoustician, asserts that whilst quality in sound is mainly dependent upon the relative amplitude of the harmonics the difference of phase makes some contribution to it.

Hence when we hear a certain vowel sound the ear appreciates the fact that it has a certain wave form as well as amplitude and wave length, for we distinguish *quality* in sounds as well as *loudness* and *pitch*.

All articulate sounds are made up of consonantal sounds and vowel sounds. The latter are continuous or can be made so to be, the former are modulations at the beginning or end of the vowel sounds. Thus the simplest articulate sound is a *syllable* which is composed of a vowel sound preceded or followed by a consonantal sound. Thus the word *PAPA* is composed of two identical syllables *PA*, each composed of an explosive consonantal sound indicated by the *P* and followed by a vowel sound *Ah* indicated by the *A*.

The vowel sound is made up of the sum of certain simple sine curve aerial vibrations differing in phase and amplitude with wave lengths or frequencies in harmonic relation.

Accordingly, if we are to transmit intelligible speech by telephone it is essential that the broad features of each syllabic sound shall be repeated at the receiving end. This means that the wave form of the current which emerges from the line at the receiving end shall not be extravagantly different from the wave form of the current at the sending end, which in turn must not differ greatly from the wave form of the air motion in front of the microphone diaphragm.

Hence the successful transmission of speech necessitates that the various constituent harmonics which combine to make the wave form of the current at the sending end of the line shall be transmitted so that they are not much displaced in relative phase or altered in relative amplitude.

4. The Reasons for the Limitations of Telephony.—We have already proved that the speed with which a simple periodic wave of electric current is transmitted along a line depends upon the wave length, and also we have shown that the rate at which the amplitude is degraded depends also upon the wave length or frequency.

The electrical disturbances of short wave lengths are more rapidly degraded and travel faster than those of longer wave length. Hence the different harmonic constituents into which we may analyse by Fourier's theorem the complex wave form of the line current representing any vowel or syllabic sound travel at different speeds and attenuate at different rates as they move along the line. If then they are synthesised by the ear aided by a receiving telephone at the end of a long line, the result may be so different from that impressed on the line at the sending end that the ear may no longer recognise the meaning of the sound. This change in the wave form of the current wave sent along the line as it travels from the sending to the receiving end is called the *distorsion* due to the line. If the distorsion is not very great the ear recognises the articulate sound to which that current wave corresponds, but if the distorsion has proceeded beyond a certain point it is no longer recognisable. The process resembles that of caricaturing a face. The caricature is a drawing in which the various features or details are not accurately drawn but distorted, some being increased or decreased more than others. If the process has not been carried beyond a certain limit we still guess for whom it is meant, but beyond that point it is unrecognisable. Hence the practical limits of telephony are found in this *distorsion* due to the line. Thus, for instance, with a certain type of cable we may obtain excellent speech transmission over twenty miles, good over thirty miles, fair or not very bad over forty miles, but extremely bad or impossible over sixty miles. In this matter we leave out of account for the moment all questions of imperfection of the transmitter, receiver, speaker's voice, or listener's ear. We assume that these are the best possible, yet nevertheless the line itself by reason of its distorsion, viz., by the unequal attenuation and velocity of simple periodic disturbances of different frequencies, imposes a limit on the distance over which good speech can be transmitted.

The improvement of telephony is therefore bound up with the improvement in the qualities of the line. We have to construct a line which shall be non-distorsional or *distorsionless*, or at least less distorsional than existing cables, and that we proceed to discuss.

5. The Improvement of Practical Telephony.—The earliest attempts to conduct telephony over long distances or through submarine cables brought prominently before telephonists the influence of the line. It soon became clear that both resistance and capacity in the line were obstacles *per se* to long distance telephony and that to improve it the resistance of the line should be kept low and its capacity small. Hence aerial lines were found better adapted for it than underground or submarine cables, and copper wire better than iron wire. It was assumed by some persons imperfectly acquainted with electrical theory that the inductance of the line was also an obstacle to telephony. A little knowledge is proverbially a dangerous thing. Electricians of the old school, educated chiefly in connection with continuous currents or with the kind of currents required in slow speed telegraphy, had acquired just sufficient information on the subject to know that the inductance of a circuit in general hinders sudden changes in the current when the electromotive force is suddenly changed. Hence it was but natural to suppose that the rapid variations of current involved in telephony would also be resisted by the inductance of the line. Inductance in the line was therefore assumed to be detrimental and to be regarded as an enemy to be overcome. Moreover, the practicians of this school had been obliged to master some elementary knowledge of the theory of the submarine telegraph cable, which will occupy us in a later chapter, and, applying this without hesitation to the more difficult and different problem of telephony, had come to the conclusion that the great remedy for the difficulties introduced by distributed capacity in the cable was to be found in decreasing the resistance. Hence an empirical rule was enunciated which endeavoured to associate good telephony with less than a certain value for the product of the capacity and resistance per mile of the telephonic cable. This rule was commonly called the "K R" law. But accumulated experience showed that it had no true scientific basis (see Oliver Heaviside's work "Electromagnetic Theory," Vol. I., p. 321, footnote). The problem of telephonic transmission is essentially different from that of telegraphic transmission.

The first physicist who endeavoured to place before practical telephonists a valid theory of telephonic transmission was Mr. Oliver Heaviside, who gave the fundamentals of the right theory in a paper on Electromagnetic Induction and its Propagation in the *Electrician* in 1887, Vol. XIX., p. 79 (see also his *Collected Papers*, Vol. II., p. 119). He also published in *The Electrician* in 1893 writings of considerable originality and power (see issues for July, August, September, 1893) on the same subject, and these were collected into a book on Electromagnetic Theory (Vol. I., pp. 409—453), published in 1893.

Meanwhile the conception that the effects of distributed capacity could be annulled by inductance or leakage had arisen in other minds.

Professor S. P. Thompson took out a British patent (No. 22,304) in 1891, in which this was clearly stated, and he followed it by other patents in 1893 (Nos. 13,064 and 15,217), in the specifications of which he describes various modes of carrying the idea out in practice. Professor S. P. Thompson also read an interesting paper on Ocean Telephony before the Electrical Congress at the Chicago World's Fair in 1893 which attracted considerable attention to the subject, in which the methods proposed in the above-mentioned specifications were described, and the general question of improving telephony and telegraphy discussed. Professor Thompson took out a fourth patent (No. 13,581) in 1894.

Mr. Heaviside's mathematical investigations had led him to see that the true obstacle to long-distance telephony was not capacity or inductance in themselves, but the unequal attenuation and velocity of the component simple periodic waves of currents travelling along the cable. We have shown in Chapter III. that the attenuation of a simple periodic wave of current travelling along a cable is dependent upon a certain quantity α, called the attenuation constant, which is a function of the primary constants of the cable R, C, L, and S and of the frequency.

The amplitude is decreased in the ratio $1 : \epsilon^{-\alpha}$ per mile of transmission. Also the speed W with which the wave is transmitted is given by $W = n\lambda = p/\beta$, where n is the frequency $p = 2\pi n$ and β is a function of R, C, L, S and p called the wave length constant. Hence waves of different frequency or wave length travel at different speeds and attenuate at different rates.

Now Mr. Heaviside showed, as proved in Chapter III., that if the primary constants of the cable were so related that $CR = LS$, or the product of the capacity and resistance per mile was numerically equal to the product of the inductance and leakage per mile in homologous units, then this inequality of attenuation and velocity was destroyed, and simple periodic waves of all frequencies would travel on such a cable with the same speed and attenuation. Also the wave form of a complex wave would travel without *distorsion*. Hence he called such a cable a *distorsionless cable*.

The reason for this name is as follows: In a distorsionless cable current waves of all frequencies travel along the cable at the same speed, viz., $1/\sqrt{CL}$, and attenuate at the same rate, viz., are reduced in amplitude by $\epsilon^{-\sqrt{SR}}$ per mile.

Therefore the different sine curve constituents or harmonics which compose a current wave representing any given vowel sound are not relatively altered as the wave proceeds. In other words, the wave form of the current is not altered in form though it may be diminished in actual size. Hence the current wave arrives at the receiving end minified or reduced in scale, but otherwise a fair copy of that which set out from the sending end. The distorsion, which is therefore a great obstacle to intelligibility, is cured by making the cable have such constants that $CR = LS$. Since in all ordinary cables the value of CR is much greater than LS, the problem of making a cable distorsionless is capable of solution in many ways. For example,

(i.) We may reduce the resistance per mile R to the necessary degree of smallness.

(ii.) We may decrease the capacity per mile C.

(iii.) We may increase the inductance per mile L.

(iv.) We may increase the leakage of the cable per mile S.

(v.) We may change two or more of the primary constants of the cable and endeavour to make the product CR as nearly equal to the product LS as possible.

All problems in engineering are, however, ultimately questions of cost, and we have to take into account also practicabilities of construction or erection.

It was long ago noticed, however, that a leak in a telegraph or telephone line was not always a detriment, and that distributed leaks sometimes appeared to improve telephonic speech.

A very interesting account is given in Mr. Heaviside's book "Electromagnetic Theory" (Vol. I., pp. 420—433, 1st ed.) of the effect of leaks and shunts upon telegraphic and telephonic transmission in certain cases. The reader would do well to refer to this account. Mr. Heaviside's work made it quite clear that inductance up to a certain degree in a telephone line, instead of being an obstacle to long-distance transmission, was the telephonist's best friend, and that what most telephonic cables required to improve speech through them was not less but more inductance. He discussed in a general manner the effect of leaks and also proved that these were in certain cases an advantage.

Mr. Heaviside, however, did not reduce his general principles to such detailed instructions as to compel the attention of practical telephonic engineers. Part of the neglect his suggestions suffered may have been due to the belief that though theoretically correct his ideas could not be economically carried into practice, and that a more practical approach to improvement was to be found in reducing the capacity and resistance of the line rather than in increasing its inductance. About the same time two other suggestions were made by Professor S. P. Thompson, as already mentioned, in a paper on Ocean Telephony read to the Electrical Congress meeting in 1893 at Chicago, at the World's Fair held in that city. In this paper he proposed, amongst other methods, the adoption of inductive leaks or shunts across the cable as a means of curing the distorsion. Again, in the same year, Mr. C. J. Reed, following one of Professor S. P. Thompson's suggestions, took out United States patents (Nos. 510,612, 510,613, December 12, 1893) for improvements in telephone lines cut up into sections by transformers. Professor S. P. Thompson urged the trial of his method in his presidential address to the Institution of Electrical Engineers of London in 1899. Other persons also either suggested or patented methods for increasing the inductance of telephone lines.

Meanwhile practical telephonic engineers confined their efforts to reducing the capacity of telephonic cables, and as far as possible consistently with economy decreased their resistance by the use of heavy high conductivity copper wires or cables.

A considerable reduction in capacity in underground cables was brought about by the introduction of paper insulated cables and cables called dry core or air insulated cables, in which the copper wire was loosely wrapped with spirals of dry paper sufficient to keep the wires insulated but the dielectric consisting in fact of air. These cables were then lead covered to keep them dry. In long-distance lines and cables the heaviest copper conductor was adopted consistent with economy.

In 1899 and 1900 two very important papers were published by Professor M. I. Pupin, in which he described a masterly investigation, both experimental and mathematical, into the properties of *loaded cables*, that is, cables having inductance coils inserted at intervals in them.

Pupin's valuable contribution to this subject was the proof given by him that a non-uniform cable having inductance coils inserted at intervals could perform the same function as a cable of equal total inductance and resistance, but with the inductance and resistance smoothly distributed, provided that the wave length of the electrical disturbance travelling along the cable extended over at least nine or ten coils.

Pupin was thus led to enunciate a suggestion at once scientifically sound and practically possible, viz., to improve telephonic transmission by loading the cable or line at equidistant intervals, small compared with a wave length, with coils of small resistance and sufficiently high inductance.

The ideas of Heaviside were thus extended into the region of practical engineering, and Pupin's loaded cable has been proved to result in a most important improvement in long-distance telephony.

It is by no means an obvious truth that a number of separate inductance coils could act in this manner to improve telephony. It has already been pointed out that when a wave of electric current or potential is travelling along a conductor, if it arrives at a place at which the inductance or capacity per unit of length suddenly changes, there will be a reflection of part of the wave just as in the case of a ray of light when passing from one medium to another of a different refractive index. Accordingly an inductance coil inserted in a uniform line causes a loss of wave amplitude by reflection, part of the wave being transmitted through the coil with diminished amplitude. If then a series of such coils are inserted at intervals in a uniform cable, a series of reflections may take place, the result of which may be to immensely diminish the amplitude of the transmitted wave.

This is always the case when the intervals between the coils are large compared with the wave length of the disturbance.

If, however, the wave length is large compared with the length of the coil intervals, then the so loaded cable acts as if the added inductance were uniformly distributed.

As this is a very important matter we shall give here an analytical proof following that originally given by Professor Pupin.

6. Pupin's Theory of the Unloaded Cable.—Pupin prefaces his mathematical treatment of the problem of the loaded cable by a discussion of the case of the propagation of periodic electric currents along a cable of ordinary type, which is essential for the sake of comparison. In the following discussion we shall follow Pupin's method with some little amplification for the sake of clearness.[1]

Let us consider a cable in the form of a loop (see Fig. 4) having an alternator A at the sending end and a receiving instrument B at the receiving end. Let the alternator generate a simple periodic electromotive force which may be represented as the real part or horizontal step of a function of the time denoted by $E\,\epsilon^{jpt}$.

Let the cable have per unit length *on each side* an inductance L, resistance R, and capacity with respect to the earth C.

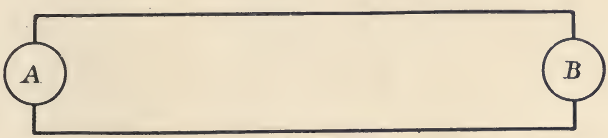

FIG. 4.

Let distance be measured from the alternator and let the distance between the alternator and receiving instrument be denoted by l. At distance x take any small length δx. Let i be the current in the cable at this point. Then the capacity of this length with respect to the earth is $C\delta x$, and the capacity with respect to a similar element in the return half of the cable is $\frac{1}{2}C\delta x$.

If then v is the potential and i the current at a distance x, the potential and current at $x + \delta x$ are $v - \frac{dv}{dx}\delta x$ and $i - \frac{di}{dx}\delta x$ respectively. Hence the fall in voltage down the element δx is

[1] Pupin's two important papers are to be found in the *Transactions of the American Institute of Electrical Engineers*, Vol. XVI., p. 93, 1899, and Vol. XVII., p. 445, 1900. The first is entitled "Propagation of Line Electrical Waves" (read March, 1899), and the second "Wave Transmission over Non-uniform Cables and Long Distance Air-Lines" (read May, 1900).

$\frac{dv}{dx}\delta x$ and the loss in current is $\frac{di}{dx}\delta x$. Hence these must be equated to the equivalent expressions, viz.,

$$\frac{dv}{dx}\delta x = L\delta x\frac{di}{dt} + R\delta x i$$

$$\frac{di}{dx}\delta x = C\delta x\frac{dv}{dt}$$

It will be noticed that Pupin considers a cable without leakage or dielectric conductance. If we differentiate the first of these equations with regard to t and the second with regard to x to eliminate v, we arrive at the equation,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} = \frac{1}{C}\frac{d^2i}{dx^2} \quad . \quad . \quad . \quad . \quad (10)$$

This is the differential equation for the propagation of an electrical disturbance in a cable having inductance L, resistance R, and capacity C per unit length of both lead and return separately, the leakage being negligible.

To formulate the boundary conditions we assume that the alternator has a resistance R_0, an inductance L_0, and that its capacity is equivalent to a capacity C_0 in series with its armature.

Suppose then that i_0 is the current in the alternator and at the sending end of the cable and that v_0 is the potential difference of the two sides of the cable at the sending end.

If then the real part of $E\epsilon^{jpt}$ represents the electromotive force of the alternator, the potential difference v_0 at the sending end of the cable is the difference between this $E. M. F.$ and the drop in voltage down the alternator circuit and the capacity in series with it.

Hence we have the equation

$$L_0\frac{di_0}{dt} + R_0 i_0 + \frac{1}{C_0}\int i_0 dt + v_0 = E\epsilon^{jpt}. \quad . \quad . \quad (11)$$

Again, if the potential difference between the ends of the cable at the receiving end is v_1 and if the receiving apparatus is equivalent to an inductive resistance (L_1, R_1) in series with a capacity C_1 and if i_1 is the current at the receiving end, we have a second boundary equation, viz.,

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt - v_1 = 0 \quad . \quad . \quad . \quad (12)$$

If the *E.M.F.* of the alternator is a simple periodic function of the time, then after a short time the current at all parts of the line will also be proportional to ϵ^{jpt}. Hence, if i varies as ϵ^{jpt}, $\frac{di}{dt}$ will be equal to jpi and $\frac{d^2i}{dt^2}$ equal to $-p^2i$.

If then we differentiate equations (11) and (12) with regard to t and make the above substitutions, we have

$$(1-C_0L_0p^2+jpC_0R_0)\,i_0+C_0\frac{dv_0}{dt}=jpC_0E\epsilon^{jpt} \quad . \quad . \quad (13)$$

If we write $\quad h_0$ for $\frac{C}{C_0}(1-C_0L_0p^2+jpC_0R_0) \quad . \quad . \quad . \quad (14)$

and $\quad D_0$ for $jpCE\epsilon^{jpt} \quad . \quad . \quad . \quad . \quad (15)$

we can transform (13) into the equation

$$C\frac{dv_0}{dt}=D_0-h_0i_0 \quad . \quad . \quad . \quad . \quad (16)$$

Now, since $C\delta x$ is the capacity of an element of length δx with regard to the earth, the capacity of a length δx with regard to a similar element in the return cable must be $\frac{C}{2}\delta x$, and hence the fall in current down the initial element δx at the sending end which is expressed by $-\frac{di_0}{dx}\delta x$ must be equal to $\frac{C}{2}\delta x\frac{dv_0}{dt}$

or
$$C\frac{dv_0}{dt}=-2\frac{di_0}{dx} \quad . \quad . \quad . \quad . \quad (17)$$

Making the substitution in (16) we have as the boundary equation at the sending end

$$-2\frac{di_0}{dx}=D_0-h_0i_0 \quad . \quad . \quad . \quad . \quad (18)$$

Similarly at the receiving end

$$2\frac{di_1}{dx}=-h_1i_1 \quad . \quad . \quad . \quad . \quad (19)$$

We have next to consider the solution of the differential equation (10). A solution applicable in the present case is

$$i=K_1\operatorname{Cos}\mu(l-x)+K_2\operatorname{Sin}\mu(l-x) \quad . \quad . \quad (20)$$

where K_1 and K_2 are functions of the time only proportional to ϵ^{jpt}.

It is easy to see that the above is a solution provided that

$$-\mu^2=C(-p^2L+jpR) \quad . \quad . \quad . \quad . \quad (21)$$

For if we differentiate (20) with regard to t and x and substitute in the original equation (10) we arrive at equation (21).

Since $-\mu^2$ is a complex quantity μ is also a complex quantity, and we can write $\mu = \beta + ja = j(a - j\beta)$.

Hence $$\beta + ja = \sqrt{Cp\,(pL - jR)} \quad . \quad . \quad . \quad (22)$$
or $\beta^2 - a^2 + j2a\beta = Cp\,(pL - jR)$.

Therefore $$\left.\begin{aligned} \beta^2 - a^2 &= LCp^2 \\ 2a\beta &= -CRp \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (23)$$

but equating the sizes of the vectors in (22) we have
$$\beta^2 + a^2 = Cp\sqrt{R^2 + p^2L^2} \quad . \quad . \quad . \quad . \quad (24)$$
and from (23) and (24) we arrive at
$$\left.\begin{aligned} a &= \sqrt{\frac{1}{2}Cp\,(\sqrt{R^2+p^2L^2} - pL)} \\ \beta &= \sqrt{\frac{1}{2}Cp\,(\sqrt{R^2+p^2L^2} + pL)} \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

Now, since $(a+x)^n = a^n + xna^{n-1}$ nearly, when x is small compared with a, and we can therefore neglect terms involving the square and higher powers of x, it follows that $\sqrt{R^2+p^2L^2} = pL + \frac{R^2}{2pL}$ when pL is large compared with R, and therefore that
$$\sqrt{R^2+p^2L^2} - pL = \frac{R^2}{2pL}.$$
Hence when pL/R is a large number we have
$$\left.\begin{aligned} a &= \frac{R}{2}\sqrt{\frac{C}{L}} \\ \beta &= p\sqrt{CL} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26)$$
and the wave velocity $W = n\lambda = \dfrac{1}{\sqrt{CL}}$.

Accordingly the attenuation constant a and the wave velocity W are independent of the frequency when the inductance per mile is large compared with the resistance per mile for moderate frequencies.

For very high frequencies pL tends to be always greater than R under any circumstances.

If $$i = K_1 \operatorname{Cos} \mu\,(l-x) + K_2 \operatorname{Sin} \mu\,(l-x) \quad . \quad . \quad . \quad (27)$$
it follows that at the sending end where $x = 0$ and $i = i_0$ we have $$-2\frac{di_0}{dx} = -2K_1\mu \operatorname{Sin} \mu l + 2K_2\mu \operatorname{Cos} \mu l \quad . \quad . \quad (28)$$

Also at the receiving end where $x = l$ and $i = i_1$ we have

$$2\frac{di_1}{dx} = -2K_2\mu \quad . \quad . \quad . \quad . \quad (29)$$

$$\left.\begin{aligned} &\text{but by (18)} & -2\frac{di_0}{dx} &= D_0 - h_0 i_0 \\ &\text{and by (19)} & 2\frac{di_1}{dx} &= -h_1 i_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

Also from (27)

$$\left.\begin{aligned} & & i_0 &= K_1 \operatorname{Cos} \mu l + K_2 \operatorname{Sin} \mu l \\ &\text{and} & i_1 &= K_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

Hence from (27), (28), (29), (30) and (31) it can easily be found that

$$K_1 = \frac{2\mu D_0}{F} \text{ and } K_2 = \frac{h_1 D_0}{F}$$

where

$$F = (h_0 h_1 - 4\mu^2) \operatorname{Sin} \mu l + 2\mu\,(h_0 + h_1) \operatorname{Cos} \mu l \quad . \quad . \quad (32)$$

Accordingly we can write (27) in the form

$$i = \frac{D_0}{F}\left\{2\mu \operatorname{Cos} \mu\,(l-x) + h_1 \operatorname{Sin} \mu\,(l-x)\right\} \quad . \quad . \quad (33)$$

and this is the complete solution of the differential equation (10).

When $h_0 = h_1 = 0$ we have

$$i = -\frac{D_0}{2\mu}\,\frac{\operatorname{Cos} \mu\,(l-x)}{\operatorname{Sin} \mu l} \quad . \quad . \quad . \quad . \quad (34)$$

In the above equations μ stands for $\beta + j\alpha$ where α is the attenuation constant and β the wave length constant. Hence the wave length is $\frac{2\pi}{\beta}$ and the attenuation for a distance x is $\epsilon^{-\alpha x}$.

Equation (33) is the general solution of the differential equation for oscillations either free or forced. If, however, the oscillations are free oscillations, then $D_0 = 0$ and hence in this last case μ must have such a value as to make $F = 0$, otherwise i would be always zero. Accordingly the condition for *free* oscillations is

$$(h_0 h_1 - 4\mu^2) \operatorname{Sin} \mu L + 2\mu\,(h_0 + h_1) \operatorname{Cos} \mu l = 0 \quad . \quad . \quad (35)$$

Suppose then that the transmitting and receiving apparatus are removed and replaced by a short circuit. This is equivalent to assuming C_0 and C_1 both to be infinitely large. Then we have $h_0 = h_1 = 0$.

The equation (35) then reduces to Sin $\mu l = 0$, and hence we must have $\mu = \frac{s\pi}{l}$ where s is some integer from 1 upwards.

Accordingly
$$-\mu^2 = -\frac{s^2\pi^2}{l^2}.$$

Referring to equation (21) we have
$$-\mu^2 = C\,(-p^2\,L + jpR) = -\frac{s^2\pi^2}{l^2} \quad . \quad . \quad . \quad (36)$$

If we write k for jp in the above equation it becomes
$$CLk^2 + CRk = -\frac{s^2\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad (37)$$

Solving this quadratic equation we have
$$k = jp = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC}\,\frac{s^2\pi^2}{l^2} - \frac{R^2}{4L^2}}. \quad . \quad . \quad (38)$$

If $2L$ is large compared with R, then
$$p = 2\pi n = \frac{s\pi}{l}\sqrt{\frac{1}{LC}} \quad . \quad . \quad . \quad . \quad (39)$$

Hence the frequencies of the possible oscillations are obtained from the equation
$$n = \frac{1}{2\pi}\,\frac{s\pi}{l}\sqrt{\frac{1}{LC}} \quad . \quad . \quad . \quad . \quad (40)$$

by giving s various integer values. The velocity of propagation of the waves is $W = \frac{1}{\sqrt{LC}}$, and hence the possible wave lengths are the values of $2l/s$ for various integer values of s, viz., $2l/1$, $2l/2$, $2l/3$, etc.

In the next place, suppose that the transmitter has no resistance or inductance but very large capacity, and that the receiving end is open. Then we must have $h_0 = 0$, and $h_1 =$ infinity. Equation (35) then reduces to Cos $\mu l = 0$ or
$$\mu = \frac{2s+1}{l}\,\frac{\pi}{2}.$$

where s is any integer.

We find then in the same manner as in the former case that
$$k = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC}\left(\frac{2s+1}{l}\,\frac{\pi}{2}\right)^2 - \frac{R^2}{4L^2}} \quad . \quad . \quad (41)$$

and if L is large compared with R
$$p = 2\pi n = \frac{2s+1}{l}\,\frac{\pi}{2}\sqrt{\frac{1}{LC}} \quad . \quad . \quad . \quad (42)$$

and the wave lengths for possible free vibrations are $4l/1$, $4l/3$, $4l/5$, etc.

7. Pupin's Theory of the Loaded Cable.—In the papers previously mentioned Pupin discusses also the mathematical theory of the cable loaded with inductance coils at equal intervals. He supposes a cable to have coils of inductance L and resistance R inserted at equal intervals and a condenser of capacity C to be connected between the earth and the junction between each coil. Also that a transmitter having inductance and resistance L_0 and R_0 with capacity C_0 is placed at A and a receiver with similar constants L_1, R_1, C_1 placed at B. A simple

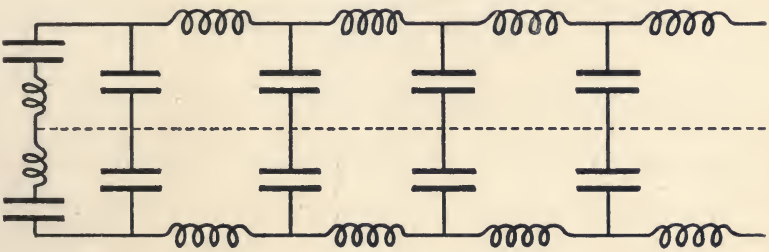

FIG. 5.—Pupin Artificial Cable.

periodic electromotive force proportional to $E\,\epsilon^{jpt}$ is in operation at the transmitter end. (See Fig. 5.)

The conductor thus consists of $2n$ coils in a loop with $2\,(n-1)$ condensers to earth between. The whole loop is thus divided into $2\,(n-2)$ component circuits. It is clear that when n becomes very large the line becomes an ordinary cable. The question then arises, under what conditions will a conductor of this kind be equivalent to a uniform cable even if n is not infinitely large? The problem of finding the time of electrical vibration of such a line is analogous to the problem of finding the free vibrations of a string loaded with weights at equal intervals which was solved by Lagrange in his "Méchanique Analytique" (Partie VI.).

Let i_1, i_2, i_3, etc., be the currents in the component circuits of the loaded line, and let v_1, v_2, v_3, etc., be the drops in potential down the condensers. Then the currents through the condensers are

$$g_1 = C\frac{dv_1}{dt}, \quad g_2 = C\frac{dv_2}{dt}, \text{ etc.,}$$

and also $g_1 = i_1 - i_2$, $g_2 = i_2 - i_3$, etc. Consider then in the first place the case of *forced oscillations* in such a loaded cable. For each mesh or circuit we can write an equation as follows:

$$\left.\begin{array}{l}
\text{1st circuit } (L_0+2L)\dfrac{di_1}{dt}+(R_0+2R)\,i_1+v_0+2v_1=E\,\epsilon^{jpt} \\
\text{2nd circuit } L\,\dfrac{di_2}{dt}+Ri_2+v_2-v_1=0 \\
(n-1)\text{th circuit } L\dfrac{di_{n-1}}{dt}+Ri_{n-1}+v_{n-1}-v_{n-2}=0 \\
n\text{th circuit } (L_1+2L)\dfrac{di_n}{dt}+(R_1+2R)i_n+v_1-2v_{n-1}=0
\end{array}\right\} \quad (43)$$

When the steady state is reached the currents will be all simple periodic currents and proportional to ϵ^{jpt}.

Hence for $\frac{d}{dt}$ we can write jp and for $\frac{d^2}{dt^2}$ we can put $-p^2$.

The above equations can then be written

$$\left.\begin{array}{l}
hi_1+g_1-0=D \\
hi_2+g_2-g_1=0 \\
hi_{n-1}+g_{n-1}-g_{n-2}=0 \\
hi_n+0-g_{n-1}=-h_1i_n
\end{array}\right\} \quad . \quad . \quad . \quad (44)$$

where
$$\left.\begin{array}{l}
h=C\,(-p^2L+jpR) \\
D=\dfrac{1}{2}jpCE\,\epsilon^{jpt}-\dfrac{1}{2}Ci_1\,(-p^2\lambda_0+jpR_0)=D_0-h_0i_1 \\
h_1=\dfrac{1}{2}\,C\,\,(-p^2\lambda_1+jpR_1) \\
\lambda_0=L_0-\dfrac{1}{p^2C_0} \quad \lambda_1=L_1-\dfrac{1}{p^2C_1}
\end{array}\right\} \quad . \quad (44a)$$

Following the analogy with the solution of the differential equation (10) in the previous section, it is clear that a solution of the equations (43) can be found in the form

$$i_m=K_1 \operatorname{Cos} 2\,(n-m)\,\theta+K_2 \operatorname{Sin} 2\,(n-m)\,\theta \quad . \quad . \quad (45)$$

If $h + 2 = 2 \operatorname{Cos} 2\,\theta$, then all the equations (44) except the first and last will be satisfied for all values of K_1 and K_2.

These two equations, which correspond to the boundary conditions in the case of the uniform cable, will be satisfied if

$$K_2 = \frac{(h_1 - 1) + \operatorname{Cos} 2\theta}{\operatorname{Sin} 2\theta} \text{ and}$$

$$K_1 = \frac{D_0 \operatorname{Sin} 2\theta}{h_0 h_1 \operatorname{Sin} 2(n-1)\theta - 4\operatorname{Sin}^2\theta \operatorname{Sin} 2n\theta + 2(h_0 + h_1)\operatorname{Sin}\theta \operatorname{Cos}(2n-1)\theta} \qquad (46)$$

We have then a solution for i_m in the form

$$i_m = \frac{[2 \operatorname{Sin}\theta \operatorname{Cos}(2n-2m+1)\theta + h_1 \operatorname{Sin} 2(n-m)\theta] D_0}{h_0 h_1 \operatorname{Sin} 2(n-1)\theta - 4\operatorname{Sin}^2\theta \operatorname{Sin} 2n\theta + 2(h_0 + h_1)\operatorname{Sin}\theta \operatorname{Cos}(2n-1)\theta} \quad . \quad (47)$$

θ is a complex angle, and hence forced oscillations of a simple periodic type on a non-uniform cable of this kind are finally simple harmonic damped oscillations.

Suppose the transmitter and receiver absent, and the cable short-circuited, then we have $h_0 = h_1 = 0$, and

$$i_m = \frac{-D_0 \operatorname{Cos}(2n-2m+1)\theta}{2 \operatorname{Sin}\theta \operatorname{Sin} 2n\theta} \quad . \quad . \quad . \quad (48)$$

In the next place let us consider the *free oscillations*.

The expression for the current given in equation (47) must hold for free as well as forced oscillations. When the oscillations are free, then the *E.M.F.* of the transmitter is zero, and hence $D_0 = 0$. Accordingly the denominator of (47) must then be zero to prevent the current vanishing.

Hence we must have in the case of free oscillations

$$h_0 h_1 \operatorname{Sin}(2n-2)\theta - 4\operatorname{Sin}^2\theta \operatorname{Sin} 2n\theta +$$
$$2(h_0 + h_1)\operatorname{Sin}\theta \operatorname{Cos}(2n-1)\theta = 0 \quad . \quad . \quad (49)$$

The first important case to consider is when the transmitter and receiver are absent, and the cable short-circuited at both ends. Then $h_0 = h_1 = 0$ and $i_m = B \operatorname{Cos}(2n - 2m + 1)\theta$. If in equations (44) we substitute the values of $g_1 = i_1 - i_2$, $g_2 = i_2 - i_3$, etc., we have

$$\left.\begin{aligned} &(h+1)\, i_1 + 0 - i_2 = D = D_0 - h_0 i_1 \\ &(h+2)\, i_2 - i_1 - i_3 = 0 \\ &(h+2)\, i_{n-1} - i_{n-2} - i_n = 0 \\ &(h+1)\, i_n - i_{n-1} - 0 = -h_1 i_n \end{aligned}\right\} \quad . \quad . \quad . \quad (50)$$

Now it is found from (49) that the value

$$i_m = B \operatorname{Cos}(2n - 2m + 1)\theta$$

is a solution of the differential equations (50) for $h_0 = h_1 = D_0 = 0$

provided that $\theta = \frac{s\pi}{2n}$, where s is some positive integer from 1 to $2n$.

Hence the most general solution for the current is then

$$i_m = \sum_{s=1}^{s=2n} B_s \operatorname{Cos}(2n-2m+1)\frac{s\pi}{2n} \quad . \quad . \quad . \quad (51)$$

Also i_m is a periodic function of the time, and may be written

$$i_m = \sum_{s=1}^{s=\infty} K_s \epsilon^{p_s t} \quad . \quad . \quad . \quad . \quad (52)$$

Hence in (51) each amplitude contains the factor ϵ^{p}

The constant p_s, which determines the period and the damping, is determined as follows:

From the second equation in (50) we have

$$(h+2)\, i_2 - i_1 - i_3 = 0, \text{ or } h+2 = \frac{i_1+i_3}{i_2}.$$

Now i_m varies as Cos $(2n - 2m + 1)\,\theta$. Hence, giving m values 1, 2, 3, successively, we have

$$i_1 : i_2 : i_3 = \operatorname{Cos}(2n-1)\,\theta : \operatorname{Cos}(2n-3)\,\theta : \operatorname{Cos}(2n-5)\,\theta$$

and
$$h+2 = \frac{\operatorname{Cos}(2n-1)\,\theta + \operatorname{Cos}(2n-5)\,\theta}{\operatorname{Cos}(2n-3)\,\theta}.$$

The quantity on the right-hand side is equal to 2 Cos 2 θ.

Hence $h = 2 \operatorname{Cos} 2\theta - 2 = -4 \operatorname{Sin}^2\theta$.

Hence for free oscillations we have

$$h = p_s^2 LC + p_s RC = -4 \operatorname{Sin}^2\theta = -4 \operatorname{Sin}^2 \frac{s\pi}{2n} \quad . \quad (53)$$

Before solving the equation (53) it is desirable to make the following substitutions:

Let L', R', and C' be the total inductance, resistance, and capacity of one half of the loaded conductor. Then

$$L = \frac{L'}{n},\ R = \frac{R'}{n},\ C = \frac{C'}{n}.$$

Let l denote the distance between the ends or half-length of a line having inductance, resistance, and capacity per unit of length denoted by u, r, and c, and let this uniform line have such values that

$$lu = L',\ lr = R',\ lc = C'.$$

Then
$$L = \frac{lu}{n},\ R = \frac{lr}{n},\ C = \frac{lc}{n}.$$

This uniform line will be called the *corresponding uniform conductor.*

We can then write the equation (53) in the form

$$\frac{l^2}{n^2}(p_s^2 uc + p_s cr) = -4 \operatorname{Sin}^2 \frac{s\pi}{2n} \quad . \quad . \quad . \quad (54)$$

where p_s takes the place of jp in equations (44a).

Solving this quadratic, we have

$$p_s = -\frac{r}{2u} \pm j\sqrt{\frac{1}{uc}\frac{4n^2}{l^2}\operatorname{Sin}^2\frac{s\pi}{2n} - \frac{r^2}{4u^2}} \quad . \quad . \quad (55)$$

or

$$p_s = -\frac{r}{2u} \pm jk_s.$$

If u is large compared with r we have

$$p_s = j\frac{2n}{l}\operatorname{Sin}\frac{s\pi}{2n}\sqrt{\frac{1}{uc}},$$

and the possible frequencies f_s are given by

$$f_s = \frac{2n}{2\pi l}\operatorname{Sin}\frac{s\pi}{2n}\sqrt{\frac{1}{uc}} \quad . \quad . \quad . \quad . \quad (56)$$

The equation for the current can then be written

$$i_m = \epsilon^{-\frac{r}{2u}t}\sum_{s=1}^{s=2n} A_s \operatorname{Cos}(2n-2m+1)\frac{s\pi}{2n}\operatorname{Cos}(k_s t - \phi) \quad . \quad (57)$$

The oscillations in the non-uniform cable have therefore the same damping coefficient as those in the equivalent uniform conductor.

The second important case is when the transmitter end of the cable is short-circuited and the receiver end is open. Then we have $h_0 = 0$, $h_1 = \infty$ and $D_0 = 0$.

Accordingly from equation (47) we find that then

$$i_m = B \operatorname{Sin}(2n-2m)\,\theta,$$

provided also that $\operatorname{Cos} 2(n-1)\,\theta = 0$ to make the denominator of (47) always zero.

Hence θ can have the values

$$\theta = \frac{2s+1}{2n-1}\frac{\pi}{2},$$

and therefore, as in the other case, the possible frequencies f_s are given by the equation

$$f_s = \frac{2n}{2\pi l}\operatorname{Sin}\frac{2s+1}{2n-1}\frac{\pi}{2}\sqrt{\frac{1}{uc}} \quad . \quad . \quad . \quad (58)$$

and the current by

$$i_m = \epsilon^{-\frac{r}{2u}t}\sum_{s=1}^{s=2n} A_{2u+1}\operatorname{Sin}(2n-2m+2)\frac{2s+1}{2n-1}\frac{\pi}{2}\operatorname{Cos}(k_{2n+1}\,t - \phi) \quad (59)$$

The angles $\frac{s\pi}{2n}$ and $\frac{2s+1}{2n-1}\frac{\pi}{2}$ have a definite physical meaning. If we consider the sth harmonic oscillation, then the current at the mth coil, which is denoted by $(i_m)_s$, is given by

$$(i_m)_s = A_s \text{ Cos } (2n-2m+1)\frac{s\pi}{2n} \text{ Cos } (k_s t-\phi).$$

The current at the m_1th coil is also

$$(i_{m1})_s = A_s \text{ Cos } (2n-2m_1+1)\frac{s\pi}{2n} \text{ Cos } (k_s t-\phi).$$

If these coils are one wave length apart, then $(i_m)_s = (i_{m_1})_s$, and $m_1 - m$ is the number of coils covered by one wave. But then we must have

$$(2n-2m+1)\frac{s\pi}{2n} = (2n-2m_1+1)\frac{s\pi}{2n} + 2\pi.$$

Hence $m_1 - n = \frac{2n}{s} = \nu_s$, and this last expression is therefore the number of coils covered by one wave length of the sth harmonic.

In the second case it can be shown in a similar manner that

$$2\left(\frac{2n-1}{2s+1}\right) = V_s.$$

Accordingly instead of $\frac{s\pi}{2n}$ and $\frac{2s+1}{2n-1}\frac{\pi}{2}$ we can write $\frac{1}{2}\frac{2\pi}{\nu_s}$.

If we consider 2π to represent the wave length and γ the angle which is the same fraction of 2π that the distance d between two consecutive coils is of a wave length, then $2\pi : \gamma = \lambda : d$, and therefore $2\pi/\nu_s = \gamma$.

Hence $\frac{1}{2}\gamma = \frac{\pi}{\nu_s} = \frac{s\pi}{2n}$ and $\text{Sin}\,\frac{1}{2}\gamma = \text{Sin}\,\frac{s\pi}{2n}$.

Now on comparing equation (40) for the frequency of free oscillations in a uniform cable with equation (56), which gives the same quantity for the non-uniform loaded cable, it is clear that if the coils are so close that $\frac{1}{2}\gamma$ is practically the same as $\text{Sin}\,\frac{1}{2}\gamma$, then the loaded line has free vibrations like the equivalent equally loaded cable. Accordingly Pupin reduced the solution of the problem to a verbal statement, which may be called Pupin's Law, as follows:

If there be a non-uniform cable line loaded with inductance coils at equal intervals, and if we consider the total inductance and resistance to be smoothly distributed along the line, then these two lines, the non-uniform and uniform lines, having the same total resistance and inductance, will be electrically equivalent for transmission purposes as long as one half of the distance between two adjacent coils expressed as a fraction of 2π taken as the wave length, is an angle so small that its sine has practically the same numerical value as that angle in circular measure.

Thus, for instance, if there are ten coils per wave the angular distance of two successive coils is 36°, and

$$\frac{1}{2}\gamma = 18^\circ = \pi/10 = 0{\cdot}31415.$$

But Sine 18° $= 0{\cdot}3090$, and therefore $\frac{1}{2}\gamma$ exceeds Sin $\frac{1}{2}\gamma$ by 1·6%.

If there are five coils per wave, then $\frac{1}{2}\gamma = 36^\circ = 0{\cdot}628$ radian;

and Sin $\frac{1}{2}\gamma =$ Sine $36^\circ = 0{\cdot}588$.

Here $\frac{1}{2}\gamma$ exceeds Sin $\frac{1}{2}\gamma$ by 6·8%.

If there are four coils per wave, then $\frac{1}{2}\gamma = 45^\circ = 0{\cdot}785$ radian, whilst Sin $\frac{1}{2}\gamma =$ Sine $45^\circ = 0{\cdot}707$, and $\frac{1}{2}\gamma$ exceeds Sin $\frac{1}{2}\gamma$ by nearly 11%.

Accordingly it is clear that if there are at least nine coils per wave the non-uniform cable is for that frequency practically equivalent to a cable in which the same inductance and resistance is smoothly distributed.

Pupin then shows in the papers mentioned that the same law holds good for forced as for free oscillations and also for a cable in which capacity is added in series with each loading inductance coil.

Pupin was therefore led to a very practical solution of the problem of constructing a telephone line which, if not absolutely distorsionless, was at least much less distorsional than ordinary unloaded lines.

Consider, for instance, the National Telephone Company's standard line, viz., a telephone cable having a resistance of 88 ohms per loop mile, an inductance of 0·001 henry per loop mile, a capacity of ·05 microfarad per loop mile, and no sensible leakage. Then $R = 88$, $C = \cdot 05 \times 10^{-6}$, $L = 0\cdot 001$, $S = 0$. Therefore for this cable $\beta = \sqrt{\frac{Cp}{2}\left\{\sqrt{R^2+p^2L^2}+Lp\right\}}$ where $p = 2\pi$ times the frequency.

As regards the frequency or range of frequency employed in telephony, the actual frequencies of the simple periodic oscillations with which articulate sounds may be analysed vary between 100 and 2,000 or so. It has been found, however, that a mean value of about 800 may be employed in the formulæ for the attenuation and wave length constants, or in round numbers we may take $p = 5{,}000$ for the case of articulate speech. Putting, then, $p = 5{,}000$ in the above formula, we have $pL = 5$, $pC = 25 \times 10^{-5}$, and

$$\sqrt{R^2+p^2L^2} = \sqrt{(88)^2+(5)^2} = \sqrt{7769} = 88\cdot 1.$$

Hence we have $\beta = \sqrt{12\cdot 5 \times 93\cdot 1 \times 10^{-5}} = 0\cdot 108$.

Therefore $\lambda = 2\pi/\beta = 58\cdot 2$ miles.

The wave length for the frequency of about 800 is therefore nearly 60 miles. Also the attenuation constant α is

$$\sqrt{12\cdot 5 \times 83\cdot 1 \times 10^{-5}} = 0\cdot 102.$$

Suppose then that the above cable has inserted in it every two miles a loading coil or inductance coil having an inductance of 0·2 heavy and negligible resistance. Then the inductance per mile becomes 0·1 henry, and for the loaded line and same frequency we have $R = 88$, $L = 0\cdot 1$, $C = 5 \times 10^{-8}$, $p = 5000$. Hence $pL = 500$ $pC = 25 \times 10^{-5}$. Therefore

$$\alpha' = \sqrt{\frac{25}{2\cdot 10^5}\left\{\sqrt{7744+25\cdot 10^4}-500\right\}} = 0\cdot 031,$$

$$\beta' = \sqrt{\frac{25}{2\cdot 10^5}\left\{\sqrt{7744+25\cdot 10^4}+500\right\}} = 0\cdot 354,$$

and $\lambda = \frac{2\pi}{\beta} = 18$ nearly.

Accordingly the effect of loading is to reduce the original attenuation constant to $\frac{1}{3}$ and the wave length in the same ratio.

Since there is one loading coil every two miles, and since the wave length of the loaded line is 18 miles, it follows that there are nine coils per wave length of the loaded line. Hence the inter-coil distance is short compared with the wave length. It is found that under these conditions the loss by reflection at each coil is not serious. If, however, the inter-coil distance were large compared with the wave length, the loss of wave energy at each reflection would be considerable. We have already shown in Chapter III. that when a wave of current passes across a point which marks a change in the constant of the line, say a sudden variation of inductance per mile, then reflection occurs, part of the wave being transmitted and part reflected. If this process is repeated at intervals long compared with the wave length the wave energy is soon frittered away. Hence if the wave form is complex and if it passes over a line loaded with lumps of inductance placed at intervals which are short compared with the fundamental wave length, but long compared with the higher harmonic wave length, then the effect will be to stop these latter or filter out the harmonics and let pass only the fundamental sine curve component.

Hence any sudden change in the capacity or inductance per mile is a source of energy loss to the transmitted wave owing to a reflection of part of the wave at this surface. An analogous effect is produced in the case of light. Suppose a tube down which a ray of light is sent. Let a partition of glass be placed in the tube. Then at this point there is a sudden change in the refractive index of the medium. Accordingly part of the wave is transmitted and part reflected back. If we were to place many plates of glass in the tube separated by intervals large compared with a wave length there would be a loss of light at each reflection, and the wave would pass through considerably weakened by the reflections.

If the thickness of the plates and of the interspaces were short compared with the wave length this would not occur.

Returning then to the above-mentioned standard cable when unloaded and loaded, it is clear that for the unloaded cable the propagation constant $P = \alpha + j\beta$ is a vector

$$P = 0{\cdot}102 + j\ 0{\cdot}108 = 0{\cdot}149\ \underline{/45^\circ}$$

nearly, whereas after loading the cable the propagation constant becomes $P' = a' + j\beta'$, or is a vector

$$P' = 0{\cdot}031 + j\,0{\cdot}354 = 0{\cdot}356\,\underline{/85^\circ}.$$

Hence the loading not only increases the size of the propagation constant, but increases its slope.

Accordingly in this cable after loading every two miles the wave length is 18 miles and there are nine coils per wave. The wave velocity $W = 1/\sqrt{CL}$ before loading is nearly 143,000 miles per second, but after loading it is reduced to 14,300 miles per second, or about 7,000 coils would be passed through per second.

Again, since Z_0, the initial sending end impedance, is equal to $\dfrac{\sqrt{R+jpL}}{\sqrt{K+jpC}}$, the result of loading the cable is to increase Z_0, and this decreases the current into the sending end for a given impressed *E.M.F.* Accordingly we see that loading the cable has the effect of producing five great improvements, as follows:

1. It increases the value of the propagation constant P both as regards size and slope.

2. It reduces the value of the attenuation constant a.

3. It reduces the wave length λ for a given frequency and also the wave velocity W.

4. It gives the cable a larger initial sending end impedance, and therefore reduces the current into the cable with a given impressed voltage.

5. It tends to unify or equalise the attenuation constants and also the wave velocities for different frequencies.

The result is that the wave form is propagated not only with less attenuation, but with less distorsion or loss of individuality, owing to the more equal attenuation and velocity of the various harmonic constituents.

8. Campbell's Theory of the Loaded Cable.—As long as the loading coils are placed at such intervals that there are eight or nine coils per wave length calculated on the assumption that the added inductance is smoothly or uniformly distributed, experience shows that the so calculated attenuation constant agrees with the results of experiment.

It is, however, necessary to establish a more general theory of the loaded line and to show how the propagation constant P, attenuation constant α, and wave length constant β can be calculated from the values of the primary constants of the line when unloaded and from the inductance and resistance of the loading coils and their distance apart, knowing of course the frequency. A general theory of the loaded line has been given by Mr. G. A. Campbell.[1]

In the paper in which he gives the theory Campbell assumes that the line is of very considerable length and is loaded at intervals of distance equal to d with coils of impedance Z.

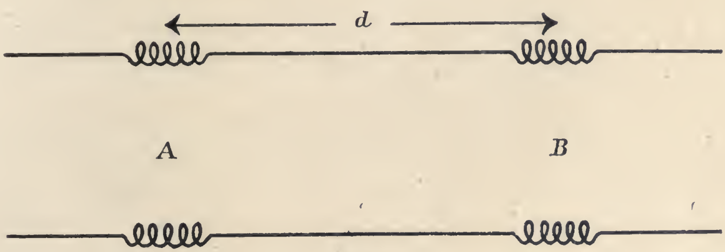

FIG. 6.

A diagrammatic representation of the line is as shown in Fig. 6.

The distance d is measured from the centre of one loading coil A to the centre of the next coil B, and the impedance Z of each coil is the sum of the two parts in the lead and return respectively.

If the line is very long we may assume that the average propagation constant is the same as the average propagation constant of one single section of length d, comprising the two half loading coils at each end and the length of line between them. The length of this section of line will always be very long compared with the length of a loading coil.

Furthermore we may assume that in the loading coil itself the current is the same at all parts of the wire composing it, and therefore the same at the centre as at the end.

We can then imagine a short circuit made at the centre of one

[1] See *Phil. Mag.*, Vol. V., p. 319, March, 1903.

coil B so that the current at the centre of that coil, which we shall call I_2, remains the same as before. Also we can imagine such an electromotive force applied between the centres of the two parts of the coil A that the current there retains the same value I_1. Hence the current in all parts of the section AB of the infinite line remains the same, and we can suppose that the parts of the line beyond B and before A are removed. We have then simply to find the average propagation constant of this finite line to solve our problem. Following a suggestion of Dr. A. E. Kennelly, we may regard this finite line in one of two ways:—

(i.) As a line of propagation constant P, which is the same as that of the unloaded line or lengths of line between the coils, which is closed at the receiving end through a receiving instrument of impedance $Z/2$.

(ii.) We may regard the line as one having an average propagation constant P', which is short-circuited at the receiving end.

In both cases the line itself is assumed to have the same initial sending end impedance Z_0.

If then the current at the sending end is I_1 and that at the receiving end is I_2, we have already shown (see Chapter III., equation (60)) that in a line of initial sending end impedance Z_0 and having a receiving instrument of impedance Z_r at the end the currents I_1 and I_2 are related as follows:

$$\frac{I_1}{I_2} = \text{Cosh } Pl + \frac{Z_r}{Z_o} \text{Sinh } Pl \quad . \quad . \quad . \quad (60)$$

In the present case the length of line is d, and the propagation constant is P, and the impedance of the supposed receiving instrument is $Z/2$.

Hence we have then

$$\frac{I_1}{I_2} = \text{Cosh } Pd + \frac{Z}{2Z_o} \text{Sinh } Pd \quad . \quad . \quad . \quad (61)$$

Again, we have shown (see Chapter III., equation (49)) that in the case of a line of length d and average propagation constant P', which is short-circuited at the receiving end, the ratio of the currents is given by

$$\frac{I_1}{I_2} = \text{Cosh } P'd \quad . \quad . \quad . \quad . \quad (62)$$

Hence this applies to the case (ii.). Equating these values of I_1/I_2, we have

$$\text{Cosh } P'd = \text{Cosh } Pd + \frac{Z}{2Z_0} \text{ Sinh } Pd \quad . \quad . \quad (63)$$

The above equation is that given by Mr. Campbell (see *Phil. Mag.*, Vol. V., p. 319, 1903), but the process of reasoning by which he arrives at it is based upon a consideration of the coefficients of reflection and transmission of each coil. His argument is much more difficult to follow than that given above, and in the opinion of the author contains one small inconsistency between his lettered diagram and the text which is extremely puzzling. Accordingly we shall not reproduce his proof verbatim here, but leave the reader to consult the original paper.

We can put Campbell's equation into another form.

If we denote $\frac{Z}{2Z_0}$ by tanh γ, as before, we have

$$\text{Cosh } P'd = \text{Cosh } Pd + \tanh \gamma \text{ Sinh } Pd \quad . \quad . \quad (64)$$

which can be written

$$\text{Cosh } P'd = \frac{\text{Cosh } (Pd+\gamma)}{\text{Cosh } \gamma} \quad . \quad . \quad . \quad (65)$$

or

$$P' = a' + j\beta' = \frac{1}{d} \text{Cosh}^{-1} \left[\frac{\text{Cosh } (Pd+\gamma)}{\text{Cosh } \gamma} \right] \quad . \quad . \quad (66)$$

We have already given the expressions for calculating the value of an inverse hyperbolic function such as $\text{Cosh}^{-1}x$ or $\text{Sinh}^{-1}x$. Hence if P, d, and γ are given, we can reduce the value of

$$\text{Cosh } (Pd+\gamma) \,/\, \text{Cosh } \gamma$$

to the form $x + jy$, and we have then for the value of $P' = a' + j\beta'$

$$P' = \frac{1}{d} \text{ Cosh}^{-1} (x+jy) \quad . \quad . \quad . \quad . \quad (67)$$

But this last is a vector quantity, and, in accordance with the proof given at the end of Chapter I., can be written in the form

$$\frac{1}{d}\text{Cosh}^{-1} (x+jy) = \frac{1}{d} \text{ Cosh}^{-1} \frac{\sqrt{(1+x)^2+y^2} \pm \sqrt{(1-x)^2+y^2}}{2}$$

$$+ j\,\frac{1}{d} \text{ Cos}^{-1} \frac{\sqrt{(1+x)^2+y^2} \pm \sqrt{(1-x)^2+y^2}}{2} \quad . \quad . \quad (68)$$

Hence, equating horizontal and vertical steps, we have for the value of the average attenuation constant α' of the loaded line the expression

$$\alpha' = \frac{1}{d} \operatorname{Cosh}^{-1} \frac{\sqrt{(1+x)^2+y^2} \pm \sqrt{(1-x)^2+y^2}}{2} \quad . \quad (69)$$

and for the average wave length constant

$$\beta' = \frac{1}{d} \operatorname{Cos}^{-1} \frac{\sqrt{(1+x)^2+y^2} \pm \sqrt{(1-x)^2+y^2}}{2} \quad . \quad . \quad (70)$$

The above formulæ lend themselves without difficulty to numerical calculation, but require some care in use. They enable us to calculate the attenuation constant for a line of certain known primary constants loaded at intervals of distance d with inductance coils of impedance Z.

On the other hand, when the coils are spaced apart so closely that the distance d does not exceed $\frac{1}{9}\frac{2\pi}{\beta}$, or one-ninth of a wave length on the loaded cable, then we can obtain just as good a value for α' and β' by considering the inductance of the coils smoothly distributed along the line.

If, however, the coils are fewer than about nine per wave length, then the resultant or true attenuation constant of the loaded line is greater than that calculated on the assumption that the added inductance is smoothly distributed over the line.

Let α' be this true attenuation constant and α'' the attenuation constant calculated from the assumption of uniformly distributed inductance, and let β' and β'' and λ' and λ'' be the corresponding wave length constants and wave lengths.

Suppose that an unloaded line has a resistance of R ohms and an inductance of L henrys per mile, the inductance being very small. Let this line be loaded with impedance coils such that the total added resistance makes the line equivalent to one having $R + R'$ ohms per mile and the total inductance equal to a line of $L + L'$ henrys per mile.

Then these values of the total resistance and inductance may be used as the R and L in the formula for calculating the attenuation and wave length constants, and they give us respectively the values of α'' and β''.

Suppose then that R' is given such a value that it is about equal to $R/2$, then the attenuation constant α'', calculated from the smoothly distributed resistance and inductance, is nearly equal to the true attenuation constant α' when there are nine coils per wave. If, however, there are less coils per wave, then α' is greater than α'' by a certain percentage, as shown in the table below.

Number of coils per wave length λ''.	Distance between coils $= d$.	Percentage by which α' exceeds α''.
9	$\lambda''/9$	Practically zero.
8	$\lambda''/8$	1%
7	$\lambda''/7$	2%
6	$\lambda''/6$	3%
5	$\lambda''/5$	7%
4	$\lambda''/4$	16%
3	$\lambda''/3$	200%

The results vary somewhat with the ratio of R'/R and L'/L In any case for less than four or five coils per wave the actual attenuation is very much greater than the attenuation calculated on the assumption that the added inductance and resistance are smoothly distributed.

If we have as few as three coils per wave the attenuation becomes so large that we may say that practically the line will not pass such a wave length at all.

Suppose that there are N impedance coils in the length of line which the current wave travels over per second; and let these coils be separated by a distance d.

Then Nd is the distance travelled by the wave per second, which is the same as its velocity, W.

But the wave velocity $W = n\lambda$, where n is the frequency and λ is the wave length. Hence we have

$$Nd = W = n\lambda,$$

or

$$N = n\frac{\lambda}{d}.$$

If we take $n = 800$ as an average value of the frequency in articulate speech, then, since experiment shows that a value of λ/d equal to 9 gives good results, we have $N = 800 \times 9 = 7{,}200$. In other words, the rate of load traversing is 7,200 coils per second.

Experiment shows also that λ/d cannot practically be less than 4 or 3. Hence $7{,}200/3 = 2{,}400$ is the highest frequency we can be concerned with in practical telephony.

For such a rate of load traversing and for such frequencies we can consider that the unequally distributed impedance at the rate of nine coils per wave gives us a line which is for all practical purposes an equally or smoothly loaded line of approximately distorsional character.

Thus, for instance, if a line having 90 ohms per mile resistance and 0·001 henry inductance and $0{\cdot}05 \times 10^{-6}$ farads capacity had inductance coils of approximately 0·2 henry inductance and 20 ohms resistance inserted every two miles, this would be equivalent to adding 10 ohms and 0·1 henry per mile; then the total resistance would be 100 ohms per mile, and the product CR per mile would be equal to 5×10^{-6}. Hence, if the insulation resistance were reduced to 20,000 ohms per mile, we should have $S = 5 \times 10^{-5}$ and $LS = 5 \times 10^{-6}$.

Such a line would be theoretically distorsionless in that all wave frequencies would travel along it at the same rate. The attenuation constant a' would be approximately equal to 0·07, whereas that of the unloaded line would be at least 0·1.

These explanations will suffice to show the very great improvement that is made in the transmission properties of a telephone line by suitable loading with impedance coils, and that, provided the insulation is not too good, we can approximate to the properties of a distorsionless line.

9. Other Methods of reducing the Distorsion of Telephone Lines.—In addition to the method above explained of loading the line with impedances, two other methods have been suggested for overcoming the distorsional quality of a telephone cable. One of these, due to Professor S. P. Thompson, consists in the insertion of inductive shunt circuits or leaks across the two members of the cable or between the line and the earth. It is clear from the explanations already given that the distorsional quality of the line depends essentially upon the excess of numerical value of the product CR over the product LS per mile of line. Hence, since CR is numerically larger than LS for any ordinary cable, we can effect the adjustment either by increasing L, as already explained, or increasing the insulation conductance S. Thus for a standard telephone line, where $R = 88$ ohms, $C = 0{\cdot}05 \times 10^{-6}$ farad, and $L = 0{\cdot}001$ henry, we should have to reduce the insulation resistance to 227 ohms per mile to bring about the necessary equalisation. This might be done by putting fifty equidistant shunts per mile, each of 10,000 ohms, between the members of the cable.

The result, however, would be to immensely increase the attenuation constant of the cable, and, although it would equalise the attenuation for different frequencies and therefore contribute to produce clearness of articulation, it would certainly decrease the volume or loudness of the sound, and loudness is as essential as clearness for intelligibility. Even if we did not lower the insulation to the full amount above given, yet the insertion of suitable non-inductive shunts across the cable does something to assist telephonic transmission.

Nevertheless it remains evident that the increase of leakage in some degree acts as an alternative method for curing distorsion in the case of telephone cables.

The subject of the effect of leakage in telephone and telegraph lines is complicated by the nature of the receiver used. The reader will, however, find some valuable information on this subject in Mr. Oliver Heaviside's book "Electromagnetic Theory," Vol. I., § 213, under the heading of "A Short History of Leakage Effects on a Cable Circuit," in which the effect of leakage on signalling speed for different types of receiving instrument is most clearly explained.

10. The Theory of the Thompson Cable.—The theory of the type of cable suggested in 1891 and 1893 by Professor S. P. Thompson for overcoming distorsion has been discussed by Dr. E. F. Roeber in an able paper following the same lines as the discussion of the Pupin cable already given.[1]

The Thompson cable consists of a lead and return conductor between which at equal intervals are connected shunt circuits

[1] See *The Electrical World and Engineer* of New York, Vol. XXXVII., pp. 440, 477, and 510, March 16th, 23rd, and 30th, 1901.

having inductance and resistance (see Fig. 7). The problem to be discussed is the right distance to place these shunts and the value of their impedance so as to effect an improvement in the distorsional qualities of the non-shunted cable.

Let the inductive shunts each have resistance R_0 and inductance L_0, and let n such shunts be bridged across in the run of the cable. Let l be the distance between the transmitter and receiver. Let the cable itself have resistance R, inductance L, and capacity C per unit of length, and suppose a simple harmonic

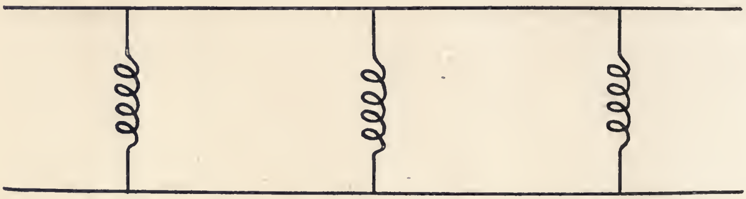

FIG. 7.—Thompson Cable with Inductive Shunts.

electromotive force denoted by the real part of $E\epsilon^{jpt}$ be operative in the transmitter.

Let $R_0 + jp\,L_0 = z_0$ and $R + jp\,L = z$.

Let i_m be the current in the line at a point between the mth and $(m+1)$th shunt at a distance x from the mth shunt.

Then at that point we can write a differential equation for the current i_m as already proved for a uniform line, viz.,

$$L\frac{d^2i_m}{dt^2}+R\frac{di_m}{dt}=\frac{1}{C}\frac{d^2i_m}{dx^2} \quad . \quad . \quad . \quad . \quad (71)$$

As already proved, this differential equation has a solution applicable in the present case in the form

$$i_m=K_1 \operatorname{Cos} \mu x+K_2 \operatorname{Sin} \mu x \quad . \quad . \quad . \quad (72)$$

where $\mu^2=-C\,(-p^2L+jpR)$.

If $\mu=\beta+ja$, then, as already shown,

$$\left.\begin{aligned} a&=\sqrt{\tfrac{1}{2}pC\,(\sqrt{R^2+p^2L^2}-pL)} \\ \beta&=\sqrt{\tfrac{1}{2}pC\,(\sqrt{R^2+p^2L^2}+pL)} \end{aligned}\right\} \quad . \quad . \quad . \quad (73)$$

The integral (72) expressing the value of i_m has to fulfil n boundary conditions at the terminations of the shunt coils.

Let g_1, g_2, g_3, etc., be the currents in the shunt coils ; then

$$g_1 = (i_0)_{x=l/n} - (i_1)_{x=0}, \text{ etc.} \qquad (74)$$

$$g_m = (i_{m-1})_{x=l/n} - (g)_{x=0} \qquad (75)$$

where $(i_0)_{x=l/n}$ stands for the current in the run of the cable in that section just *before* the first shunt close up to the junction of the shunt and $(i_1)_{x=0}$ stands for the current in the section *after* the first shunt at a point close to the junction of the shunt.

Let v_1, v_2, v_3, etc., be the potentials at one end of the shunts, and let v_1', v_2', v_3', be the potentials at the other ends. Then $v_1 - v_1'$, etc., are the drops in potential down the shunts.

Let V_m stand for the potential in the run of the cable at any point between the m^{th} and $(m+1)^{\text{th}}$ shunt.

Then V_m satisfies a differential equation of the type of (71), and this has an integral like (72), viz.,

$$V_m = N_1 \operatorname{Cos} \mu x + N_2 \operatorname{Sin} \mu x \qquad (76)$$

also

$$(V_m)_{x=l/n} = v_m = (V_{m+1})_{x=0} \qquad (77)$$

using the same notation as in the case of the currents. Likewise

$$v_m - v_m' = R_0 g_m + L_0 \frac{dg_m}{dt} \qquad (78)$$

But when the currents and potentials are steady $v_m - v_m'$ varies as $A\epsilon^{jpt}$.

Hence

$$g_m = \frac{2v_m}{R_0 + jpL_0} = \frac{2v_m}{z_0} \qquad (79)$$

Now it is clear that $C\dfrac{dV_m}{dt} = -\dfrac{di_m}{dx}$, and hence from (72) and (76)

$$\frac{dN_1}{dt} = -\frac{\mu}{C}K_2, \quad \frac{dN_2}{dt} = \frac{\mu}{C}K_1.$$

Therefore $v_m = N_1$, and $v_{m+1} = N_1 \operatorname{Cos} \dfrac{\mu l}{n} + N_2 \operatorname{Sin} \dfrac{\mu l}{n}$.

And

$$K_1 = \frac{jpC}{\mu \operatorname{Sin} \frac{\mu l}{n}} \left(v_{m+1} - v_m \operatorname{Cos} \mu \frac{l}{n} \right)$$

$$K_2 = -\frac{jpC}{\mu} v_m.$$

Therefore, substituting these values of K_1 and K_2 in (72), we have

$$i_m = \frac{jpC}{\mu \operatorname{Sin} \frac{\mu l}{n}} \left\{ v_{m+1} \operatorname{Cos} \mu x - v_m \operatorname{Cos} \mu \left(\frac{l}{n} - x \right) \right\} \qquad (80)$$

This equation is correct only from $m = 1$ to $m = n - 1$, but for i_0 and i_n, viz., the currents in the end sections, we have to develop special formulæ. It is not difficult to see that the currents in the transmitter and receiver sections are

$$i_0 = \frac{jpC}{\mu \text{ Sin } \frac{\mu l}{2n}} \left\{ v_1 \text{ Cos } \mu x - \frac{1}{2} E \epsilon^{jpt} \text{ Cos } \mu \left(\frac{l}{2n} - x \right) \right\} \quad . \quad (81)$$

$$i_n = - \frac{jpC}{\mu \text{ Sin } \frac{ul}{2n}} \text{ Cos } \mu \left(\frac{l}{2n} - x \right) v_n \quad . \quad . \quad . \quad (82)$$

We can now write the boundary equations.

Let
$$\sigma = - \frac{2\mu \text{ Sin } \frac{\mu l}{n}}{jpC\, z_0} - 4 \text{ Sin}^2 \frac{\mu l}{2n} \quad . \quad . \quad . \quad (83)$$

or
$$\sigma + 2 = - \frac{2\mu \text{ Sin } \frac{\mu l}{n}}{jpC\, z_0} + 2 \text{ Cos } \frac{\mu l}{n} \quad . \quad . \quad . \quad (84)$$

Then the boundary equations are as follows:

$$\left. \begin{aligned} &(\sigma + 3)\, v_1 + v_2 = E\, \epsilon^{jpt} \text{ Cos } \frac{\mu l}{2n} \\ &(\sigma + 2)\, v_2 - v_1 - v_3 = 0 \\ &(\sigma + 2)\, v_m - v_{m-1} - v_{m+1} = 0 \\ &(\sigma + 2)\, v_{n-1} - v_{n-2} - v_n = 0 \\ &(\sigma + 3)\, v_n - v_{n-1} \end{aligned} \right\} \quad . \quad . \quad . \quad (85)$$

If the transmitting and receiving instruments have no impedance, then $h_0 = h_1 = 0$, $h = \sigma = - 4 \text{ Sin}^2 \theta$, and let

$$D_0 = E\, \epsilon^{jpt} \text{ Cos } \frac{\mu l}{2n}.$$

Then we have

$$v_m = \frac{1}{2} \text{ Cos } \frac{\mu l}{2n} E\, \epsilon^{jpt} \frac{\text{Sin } (2n - 2m + 1)\, \theta}{\text{Sin } 2n\theta \text{ Cos } \theta} \quad . \quad . \quad (86)$$

as an equation which determines the potential at the end of a shunt coil.

The question then arises how far apart must or may the inductive shunts be placed in order that the Thompson cable may be electrically equivalent to a certain uniform line called the equivalent conductor. In the case of the Pupin loaded line the equivalent conductor is a conductor having the same total inductance and resistance as the loaded line, but spread uniformly, and we have shown that if the angular distance between the coils is γ on the same scale that the wave length is 2π, then as long as $\frac{1}{2}\gamma$ is not very different from Sin $\frac{1}{2}\gamma$ a line so loaded may for transmission purposes be replaced by the equivalent uniform line. In the case of the Thompson line we have, however, first to define what we mean by the "corresponding uniform conductor."

Let us consider the equation (83) by which σ is determined, we have

$$\sigma = -4\,\mathrm{Sin}^2\,\theta = -\frac{2\mu\,\mathrm{Sin}\,\mu\,\frac{l}{n}}{jpCz_0} - 4\,\mathrm{Sin}^2\,\mu\frac{l}{2n} \quad . \quad . \quad (87)$$

If $\mu = \beta + ja$ where β is large compared with a, then the wave length on the unloaded uniform wire is $\lambda = 2\pi/\beta$, and the angular distance between the consecutive coils for the wave length λ is γ, where

$$\gamma = \frac{l}{n}\frac{2\pi}{\lambda} = \frac{l}{n}\beta \quad . \quad . \quad . \quad . \quad (88)$$

If then γ is so small that Sin $\gamma = \gamma$ nearly, the above equation for σ can be written

$$-4\,\mathrm{Sin}^2\,\theta = -\frac{l^2\mu^2}{n^2}\left(1 + \frac{2}{jpCz_0}\,\frac{n}{l}\right) = -\frac{l^2\mu_1^2}{n^2} \quad . \quad . \quad (89)$$

where
$$\mu_1^2 = \mu^2\left(1 + \frac{2}{jpCz_0}\,\frac{n}{l}\right).$$

Hence we get
$$\mathrm{Sin}\,\theta = \frac{1}{2}\,\frac{l}{n}\mu_1.$$

If we insert in the above equation the values of μ and z_0, viz., $z_0 = R_0 + jpL_0$ and $\mu = \sqrt{\{-C(-p^2L + jpR)\}}$ we reach an equation,

$$-\mu_1^2 = C\,(-p^2L_1 + jpR_1)\,. \quad . \quad . \quad . \quad (90)$$

in which

$$L_1 = L - \frac{2n}{p^2CL}\cdot\frac{RR_0 + p^2LL_0}{R_0^2 + p^2L_0^2} \quad . \quad . \quad . \quad (91)$$

$$R_1 = R - \frac{2n}{CL}\cdot\frac{L_0R - LR_0}{R_0^2 + p^2L_0^2} \quad . \quad . \quad . \quad (92)$$

Suppose then that we have a uniform line the inductance and resistance of which per unit of length are L_1 and R_1 as given by the above equations, its capacity per unit of length being C, then this line is the "corresponding uniform line" with which the Thompson cable has to be compared.

We can now prove the equivalence of the Thompson loaded line to the equivalent uniform line defined as above.

If $\mu_1 = \beta_1 + j\alpha_1$ we have $\beta_1 = \frac{2\pi}{\lambda_1}$ where λ_1 is the wave length for the frequency $p/2\pi$ in the corresponding uniform conductor just defined. If λ_1 is represented as an angle 2π, then the angular distance between two successive shunts is γ_1, such that

$$\gamma_1 = \frac{l}{n}\frac{2\pi}{\lambda_1} = \frac{l}{n}\beta_1 \quad . \quad . \quad . \quad . \quad (93)$$

If we assume $\frac{1}{2}\gamma_1$ is so small that $\frac{1}{2}\gamma_1 = \text{Sin}\ \frac{1}{2}\gamma_1$ nearly, and also $\frac{l}{2n}\beta_1$ so small that $\epsilon^{\frac{l}{2n}\beta_1} = 1 + \frac{l}{2n}\beta_1$, we get $\theta = \frac{1}{2}\frac{l}{n}\mu_1$, and our equation (86) for the value of v_m on the Thompson line becomes identical with the value for a corresponding uniform cable as above defined.

Accordingly we can summarise the results by saying that—

A loaded cable of the Thompson type with inductive shunts at equal intervals is equivalent to its corresponding uniformly loaded cable characterised by inductance and resistance per unit of length as defined in equations (91) and (92) as long as the sine of half the angle denoting distance between two consecutive shunts is not sensibly different from the angle itself, the angle being reckoned on such a scale that the wave length for the frequency considered is equal to 2π. We see then that the rule for spacing the shunts in a Thompson cable is verbally the same as the rule for spacing the inductance coils in a Pupin cable.

The difference between the Pupin and Thompson methods is, however, that in the former we increase the effective inductance of the cable to cure distorsion and necessarily increase its resistance as well, which resistance increase we must, however, keep as small as possible. In the latter we reduce the resistance of the cable and necessarily reduce its effective inductance as well. This reduction in inductance must, however, be kept as small as possible. Hence the necessity for the use of *inductive* shunts and not inductionless shunts.

We can obtain an expression for the average attenuation of the Thompson loaded line very much on the same principles that we have obtained one for the Pupin line in § 8. We can consider the Thompson line to be made up of a series of sections, each of which consists of a double length d of plain line having a propagation constant P and a coil connected across the end having an impedance Z_r.

Let us suppose that the *P.D.*'s across the ends of these inductive shunts are denoted by V_1, V_2, V_3, etc., then each section may be regarded as a short line of length d having a receiving instrument of impedance Z_r across its far end and a *P.D.* across this coil represented by V_{n+1}, whilst the *P.D.* across the sending end is V_n. Then from the expressions given in Chapter III., if V_1 is the sending end *P.D.* and I_1 the sending end current and Z_1 the final sending end impedance and V_2, I_2 and Z_2 the corresponding quantities for the receiving end, we have

$$V_1=I_1Z_1, \quad V_1=I_2Z_2, \quad I_2=\frac{V_2}{Z_r}.$$

Hence $$\frac{I_2}{I_1}=\frac{Z_1}{Z_2} \text{ and } \frac{V_2}{V_1}=\frac{Z_r}{Z_2}.$$

Again, since the sending end voltage for the second section is equal to the *P.D.* at the ends of the shunt coil terminating the first section, we have for the second section

$$\frac{V_3}{V_2}=\frac{Z_r}{Z_2} \text{ or } \frac{V_3}{V_1}=\left(\frac{Z_r}{Z_2}\right)^2.$$

In the same way we can prove that

$$\frac{V_r}{V_1}=\left(\frac{Z_r}{Z_2}\right)^{n-1} \quad . \quad . \quad . \quad . \quad (94)$$

But $$V_1=I_1Z_1 \text{ and } V_n=I_nZ_r.$$

Hence $$\frac{I_n}{I_1}=\frac{V_nZ_1}{V_1Z_r}=\frac{Z_1}{Z_r}\frac{(Z_r)^{n-1}}{Z_2},$$

or $$\frac{I_n}{I_1}=\frac{Z_1}{Z_2}(Z_r)^{n-2} \quad . \quad . \quad . \quad . \quad (95)$$

But $\frac{I_n}{I_1}=\epsilon^{-P'nd}$ where P' is the average propagation constant of the Thompson line.

Again by equations (61) and (62) in Chapter III.

$$\frac{Z_1}{Z_2}=\frac{Z_0}{Z_0 \text{ Cosh } Pd+Z_r \text{ Sinh } Pd} \quad . \quad . \quad . \quad (96)$$

We have then

$$\epsilon^{-P'nd} = \frac{Z_0(Z_r)^{n-2}}{Z_0 \operatorname{Cosh} Pd + Z_r \operatorname{Sinh} Pd} \quad . \quad . \quad (97)$$

If then we are given Z_0, Z_r, P, n and d, we can calculate

$$\epsilon^{-P'nd} = \operatorname{Cosh} P'nd - \operatorname{Sinh} P'nd.$$

If, therefore, we denote by α' the equivalent attenuation constant of the Thompson line, we can say that $\epsilon^{-\alpha'nd}$ is equal to the real part of the expression on the right-hand side of equation (97), and therefore that $-\alpha'nd$ is equal to its Napierian logarithm. We can then find α' in terms of the given quantities.

The arithmetic, however, would be tedious.

The general result of experimental investigation on the matter as far as it has gone goes, however, to show that for a given amount of iron and copper in the form of impedance coils it results in a less attenuation constant to employ them in the Pupin fashion as coils in series rather than in the Thompson fashion as coils in parallel.

11. Other proposed Methods of constructing Distorsionless Cables.—In addition to the methods comprising the addition of inductance in series with the line and that

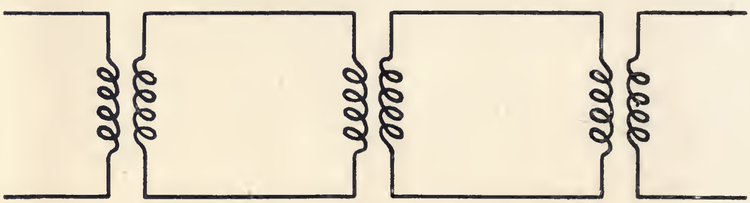

FIG. 8.—Thompson Transformer Cable.

of inserting inductive shunts across the line, a third method was proposed by Professor S. P. Thompson in his paper on Ocean Telephony in 1893, consisting in cutting up the cable into sections inductively connected by tranformers (see Fig. 8).

This plan was also proposed by Mr. C. J. Reed in 1893,[1] although it had been previously mentioned and specified by Professor S. P. Thompson.

If these transformers have a 1 : 1 ratio of transformation, or

[1] See United States Patent Specification of C. J. Reed, Nos. 510,612 and 510,613.

indeed any other ratio, they are electrically equivalent to the addition of inductance in series with the line associated with inductive shunts across the line. Accordingly it has been proved mathematically by Dr. E. F. Roeber that such a transformer cable as in Fig. 8 is electrically equivalent to the arrangement shown in Fig. 9.[1] He has also proved mathematically by an analysis on the lines of that already given for the Pupin and the Thompson cable that the transformer cable can be replaced by a certain line having a uniform distribution of inductance, resistance and capacity called the "corresponding uniform line" provided that the intervals between the transformers are short

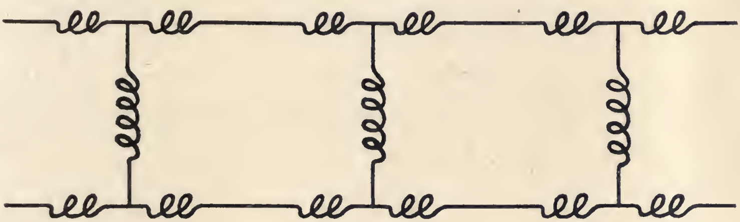

FIG. 9.

compared with the wave length, or if that interval is denoted by an angle γ on the same scale that the wave length is denoted by 2π, then the transformer line differs from the "corresponding uniform line" to the same extent that $\text{Sin}\,\frac{1}{2}\gamma$ differs from $\frac{1}{2}\gamma$.

It is hardly necessary to give the full analytical theory of this transformer cable, as the writer is not aware that it has yet been employed in practice, but the reader can be referred to Dr. Roeber's article for additional information.

The type of loaded cable suggested by Pupin has, however, come into extensive use, and in a later chapter we shall describe some of the results of practical experience and the confirmation they give of the above theory.

[1] See *The Electrical World and Engineer* of New York, Vol. XXXVII., p. 510, 1910. Dr. Roeber calls this transformer line a Reed-cable.

CHAPTER V

THE PROPAGATION OF CURRENTS IN SUBMARINE CABLES

1. The Differential Equation expressing the Propagation of an Electric Current in a Cable.— If we assume a cable to have resistance R, inductance L, capacity C, and leakance S, all per unit of length, and if the current at any distance x from the origin at any time t is i and the potential is v, then we have seen (see Chapter III.) that we can express the state of affairs at that point x by two differential equations, viz.,

$$\left.\begin{aligned}\frac{dv}{dx}&=L\frac{di}{dt}+Ri\\ \frac{di}{dx}&=C\frac{dv}{dt}+Sv\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

The first of these equations expresses the fact that the fall in potential down an element of the cable is due to the combined effect of resistance and reactance or inductance, and the second that the change in the value of the current in passing along an element of the cable is due to the combined effect of capacity and of leakage. If we differentiate the first equation with regard to x and the second with regard to t and eliminate $\frac{d^2i}{dx\,dt}$ we obtain

$$\frac{d^2v}{dx^2}=CL\,\frac{d^2v}{dt^2}+(RC+SL)\,\frac{dv}{dt}+RSv=0 \quad . \quad . \quad (2)$$

and a similar equation in i can also be reached by reversing the order of the differentiations. The above differential equation (2) is of the type

$$\frac{d^2y}{dx^2}=A\,\frac{d^2y}{dt^2}+B\,\frac{dy}{dt}+Cy=0 \quad . \quad . \quad . \quad (3)$$

The full discussion of this equation would lead us into mathematical questions of an advanced nature. Suffice it to say that it can be satisfied by many functions of x and t. Thus for instance it can be satisfied by a function of the form $y = \epsilon^{-at} \operatorname{Sin} bx$, provided there are certain relations between the constants.

Thus if $v = \epsilon^{-at} \operatorname{Sin} bx$, and we find the values of $\frac{d^2v}{dt^2}$, $\frac{dv}{dt}$ and $\frac{d^2v}{dx^2}$ from the above expression and substitute them in (2), we have

$$CLa^2-(RC+LS)\,a+RS+b^2=0 \quad . \quad . \quad . \quad (4)$$

Solving the above quadratic equation we obtain

$$a=\frac{1}{2}\left(\frac{R}{L}+\frac{S}{C}\right) \pm \sqrt{\frac{1}{4}\left(\frac{R}{L}-\frac{S}{C}\right)^2-\frac{b^2}{CL}} \quad . \quad . \quad (5)$$

The quantity b is determined by the distribution of potential along the origin of time or when $t = 0$. If then we take a point at a unit of distance from the origin or take $x = 1$, we have $v = \operatorname{Sin} b$ or $b = \operatorname{Sin}^{-1} v$. In other words, b is the inverse sine of the potential at a unit of distance from the sending end at the instant from which time is reckoned.

Suppose we assume an initial distribution such that the potential varies along the cable according to a simple sine law of distribution. Then $2\pi/b$ is the wave length. If then the constants of the cable are such that $\frac{1}{4}\left(\frac{R}{L}-\frac{S}{C}\right)^2$ is greater than $\frac{b^2}{LC}$ the quantity under the square root sign in (5) is real, and the quantity a is therefore real, and the potential at any point in the cable dies away exponentially or according to a geometric law of decrease, but without oscillations. If, however, $\frac{1}{4}\left(\frac{R}{L}-\frac{S}{C}\right)^2$ is less than $\frac{b^2}{LC}$ the value of a is a complex quantity, viz.,

$$a=\frac{1}{2}\left(\frac{R}{L}+\frac{S}{C}\right) \pm jq \quad . \quad . \quad . \quad . \quad (6)$$

where q^2 stands for $\quad \frac{b^2}{LC}-\frac{1}{4}\left(\frac{R}{L}-\frac{S}{C}\right)^2$

Hence $\quad v=\epsilon^{-\frac{1}{2}\left(\frac{R}{L}+\frac{S}{C}\right)t} \operatorname{Sin} bx\,(\operatorname{Cos} qt-j \operatorname{Sin} qt)$,

which indicates that there is at any fixed point in the cable a decadent oscillation of potential with time, the potential ultimately becoming zero.

Another solution of the differential equation (2) more applicable in the case with which we are concerned is

$$v=A\epsilon^{-\frac{1}{2}\left(\frac{R}{L}+\frac{S}{C}\right)t}\,\mathrm{Sin}\,(bx\pm qt) \quad . \quad . \quad . \quad (7)$$

This represents a damped or decaying oscillation of wave length $2\ \pi/b$ propagated with a velocity q/b along the cable.

If the constants of the cable have such relation that $\frac{R}{L}-\frac{S}{C}=0$, that is if $CR = LS$, or if the cable is distorsionless, then the quantity a is always real and $q^2 = \frac{b^2}{LC}$, or $\frac{q}{b} = \frac{1}{\sqrt{LC}}$, that is, the oscillations of all frequencies are propagated with the same velocity, $1/\sqrt{LC}$.

If we assume that v is a simple periodic quantity and can be represented by the real part of $A\epsilon^{jpt}$, then $\frac{dv}{dt}=jpv$ and $\frac{d^2v}{dt^2} = -p^2v$, so that the differential equation (2) then takes the form

$$\frac{d^2v}{dx^2}=\left\{(RS-CLp^2)+j\,(RC+SL)p\right\}v$$

or

$$\frac{d^2v}{dx^2}=(S+jpC)\,(R+jpL)v \quad . \quad . \quad . \quad . \quad (8)$$

This is the equation we have already fully discussed in dealing with the propagation of currents in telephone cables where we can assume that v varies in accordance with some function of the time which by Fourier's theorem can be resolved into the sum of a number of simple periodic terms.

In dealing with the problem of the submarine telegraph cable, however, the differential equation can be somewhat simplified as in the next section.

2. The Discussion of the Telegraph Equation.—In telegraphic signalling the changes of current or potential at the sending end are generally so slow and the inductance of the cable so small that the quantity pL or $2\pi n$L, where n is the frequency, is small compared with the resistance R. Also the leakage is so small that S is negligible. Hence the general equation (2) reduces to

$$\frac{d^2v}{dx^2}=RC\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation is called the "telegraph equation." It first presented itself in connection with a problem on the conduction of heat in a bar, but was established as the fundamental differential equation in the theory of the telegraphic cable by Lord Kelvin (then Professor William Thomson) in a celebrated classical paper "On the Theory of the Electric Telegraph" communicated to the Royal Society of London in May, 1855 (see "Mathematical and Physical Papers of Lord Kelvin," Vol. II., article lxxiii., p. 61).

The discussion of this equation as given by Lord Kelvin is not exactly suited for an elementary treatise, but it has been simplified, especially by the late Professor Everett in a volume on electricity and magnetism forming part of a revised edition of Deschanel's "Natural Philosophy." We shall follow the general method of this latter treatment.

Consider the equation

$$\frac{d^2v}{dx^2}=RC\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The following are two particular solutions:—

$$v=B+Dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$v=A\epsilon^{-k\beta^2t}\,\mathrm{Sin}\,\beta x \quad . \quad . \quad . \quad . \quad (12)$$

where $k = 1/RC$ and A, B, and C are constants.

It is clear that (11) satisfies (10). Also, if (12) is differentiated twice with regard to x it gives $-\beta^2v$, and if differentiated with regard to t and multiplied by $RC = 1/k$ we have also $-\beta^2v$. Therefore (12) is a solution of (10) subject to $k = 1/RC$. A precisely similar equation to (10) presents itself in considering the conduction of heat along a bar and also the diffusion of salt through a tube of water or other solvent.

Thus if we have a metal bar of unit cross section and thermal conductivity k, composed of a material of specific heat c, and if we consider a small section of length δx, and if the temperature on one side of the section is v and on the other $v + \frac{dv}{dx}\,\delta x$, the temperature gradient down the section is $\frac{dv}{dx}$ and the rate of flow of heat into the section is $k\frac{dv}{dx}$. Hence the rate of accumulation of heat in the section is expressed by $\frac{d}{dx}\left(k\frac{dv}{dx}\right)\delta x$. But this can also be expressed by $c\delta x\frac{dv}{dt}$, where $c\delta x$ is the amount of heat required to raise the section δx one degree in temperature. Equating these two identical expressions we have

$$\frac{d^2v}{dx^2}=\frac{c}{k}\frac{dv}{dt}.$$

Again, if we have a tube of solvent of unit section and consider the diffusion of some salt along it, we have a precisely similar equation, only in this case k stands for the diffusivity of the salt and c for the mass of salt required to produce unit concentration per cubic unit of volume of the solvent. Lastly, the same type of differential equation comes to notice in considering the gradual penetration of an electric current into a conductor, since all the above cases, propagation of potential along a submarine cable, salt diffusion, and thermal conduction are really cases of diffusion of electricity, matter, or heat.

3. The Theory of the Submarine Cable.—Suppose a cable of length l to have its distant or receiving end earthed and to have a distribution of potential made along it which is represented by the equation

$$v=A\ \mathrm{Sin}\frac{m\pi x}{l} \quad . \quad . \quad . \quad . \quad (13)$$

This means that the potential at the sending end ($x = 0$) is to be zero, and that at the receiving end ($x = l$) is to be zero, and that a maximum potential $v = A$ exists at some intermediate point.

Let this potential distribution be left to itself, then the first question is what function of the distance x and the time t will represent the distribution after the lapse of any stated time. It must be such a function that it satisfies the equation

$$\frac{d^2v}{dx^2}=RC\frac{dv}{dt} \text{ or } \frac{d^2v}{dx^2}=\frac{1}{k}\frac{dv}{dt}.$$

Also it must satisfy the boundary conditions; that is, have a zero value both for $x = 0$ and $x = l$ and a value A Sin $\frac{m\pi x}{l}$ for $t = 0$. Such a function is

$$v = A\,\epsilon^{-m^2ut} \operatorname{Sin} \frac{m\pi x}{l} \quad . \quad . \quad . \quad . \quad (14)$$

For it obviously reduces to (13) when $t = 0$ and it is zero when $x = 0$ or $x = l$. If twice differentiated with regard to x it becomes $-\frac{m^2\pi^2}{l^2} v$, and if differentiated with regard to t it yields $-m^2uv$. Hence if $u = \frac{\pi^2}{RCl^2}$ the expression (14) satisfies the differential equation (10).

Accordingly it is seen that the expression for the distribution of potential at zero time, viz.,

$$v = A \operatorname{Sin} \frac{m\pi}{l} x \quad . \quad . \quad . \quad . \quad (15)$$

is changed by lapse of time t to the expression

$$v = A\,(\epsilon^{-m^2ut}) \operatorname{Sin} \frac{m\pi x}{l} \quad . \quad . \quad . \quad . \quad (16)$$

and both of these satisfy all the conditions; provided $u = \frac{\pi^2}{RCl^2}$.

If we assume *any* distribution of potential it must be capable of being represented by a single valued curve, because the potential can only have one value at any one point at the same instant. Now such a curve can be resolved by the Fourier analysis into the sum of a number of simple periodic or sine curves of different amplitude and phase. Hence if we can express in the form of a Fourier series the initial distribution of potential, then after the lapse of a time t this distribution if left to subside will be changed into one which is expressed by multiplying each term of the above Fourier series, which is a term of the form A Sin $\frac{m\pi x}{l}$, by an exponential factor of the form ϵ^{-m^2ut}, since each term of the original and each term of the so altered series satisfies the differential equation and also the boundary conditions.

For the same cable the quantity $u = \frac{\pi^2}{RCl^2}$ has a constant value, and hence the exponential factors for the different terms will have the same value at times t which are inversely as m^2 or directly proportional to the square of the wave length λ because the quantity $\frac{m\pi}{l}$ must be equal to $\frac{2\pi}{\lambda}$. Accordingly the terms representing waves of short wave length die away more quickly than long ones.

Suppose then that at the sending end of the cable we apply one pole of a battery and raise the end to a potential V, the receiving end remaining connected to earth. There will after a time be a final distribution of potential gradually diminishing from V at the sending end to zero at the receiving end, and the

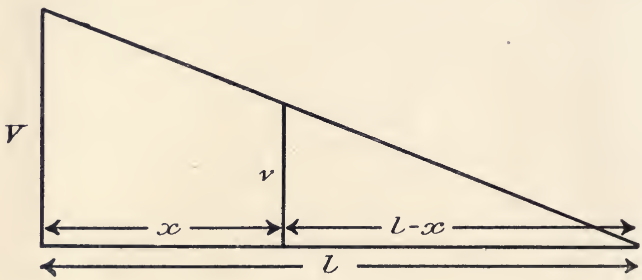

FIG. 1.

potential at any distance x from the sending end will be represented by the expression

$$v=V\frac{l-x}{l} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For this expression (17) represents a potential gradient in the form of a straight line. (See Fig. 1.)

If this steady state is altered by putting the sending end to earth at the time $t = 0$, then the potential becomes zero at the sending end or $v = 0$ for $x = 0$, and at every other point it is represented by $v = V\frac{l-x}{l}$.

To find the subsequent distribution we have to expand the last expression into a series of sine terms and find the coefficients.

$$\text{If} \quad y=\frac{l-x}{l}=A_1 \operatorname{Sin}\frac{\pi x}{l}+A_2 \operatorname{Sin}\frac{2\pi x}{l}+\text{etc.}+A_m \operatorname{Sin}\frac{m\pi x}{l} \quad . \quad (18)$$

We proceed to find the values of the co-efficients A_1, A_2, . . . A_m in the manner already explained in Chapter IV. Multiply both sides of the expression by $\text{Sin}\,\frac{m\pi x}{l}\,\delta x$ and take the average value of each term between $x = 0$ and $x = 2l$. Then all products on the right hand side vanish except one, because the average value of such an expression as $\text{Sin}\,n\,\theta\,\text{Sin}\,m\,\theta$ is zero when taken over one complete period.

Hence we have left

$$\frac{1}{2l}\int_0^{2l}\frac{l-x}{x}\,\text{Sin}\,\frac{m\pi x}{l}\,\delta x=\frac{1}{2l}\int_0^{2l}A_m\,\text{Sin}^2\frac{m\pi x}{l}\,\delta x\,. \qquad . \quad (19)$$

Now $$\int\frac{l-x}{l}\,\text{Sin}\,\frac{m\pi x}{l}\,\delta x=\int\text{Sin}\,\frac{m\pi x}{l}\,\delta x-\int\frac{x}{l}\,\text{Sin}\,\frac{m\pi x}{l}\,\delta x$$

but $$\int\text{Sin}\,\frac{m\pi x}{l}\,\delta x=-\frac{l}{m\pi}\,\text{Cos}\,\frac{m\pi x}{l}$$

also $$\int x\,\text{Sin}\,\frac{m\pi x}{l}\,\delta x=-\frac{l^2}{m^2\pi^2}\,\text{Sin}\,\frac{m\pi x}{l}-\frac{lx}{m\pi}\,\text{Cos}\,\frac{m\pi x}{l}.$$

Hence

$$\int\frac{l-x}{l}\,\text{Sin}\,\frac{m\pi x}{l}\,\delta x=-\frac{l}{m\pi}\,\text{Cos}\,\frac{m\pi x}{l}-\frac{l}{m^2\pi^2}\,\text{Sin}\,\frac{m\pi x}{l}+\frac{x}{m\pi}\,\text{Cos}\,\frac{m\pi x}{l}$$

$$=-\frac{(l-x)}{m\pi}\,\text{Cos}\frac{m\pi x}{l}-\frac{l}{m^2\pi^2}\,\text{Sin}\,\frac{m\pi x}{l}.$$

The value of this last integral between the limits $x = 0$ and $x = 2l$ is $\frac{2l}{m\pi}$.

Again, the integral $$\int\text{Sin}^2\,\frac{m\pi x}{l}\,\delta x=\int\left(\frac{1}{2}-\frac{1}{2}\,\text{Cos}\,\frac{2m\pi x}{l}\right)\delta x$$

or $$\int\text{Sin}^2\frac{m\pi x}{l}\delta x=\frac{x}{2}-\frac{l}{4m\pi}\,\text{Sin}\,\frac{2m\pi}{l}\,x$$

and the value of this between the limits $x = 0$ and $x = 2l$ is l.

Hence the result of multiplying both sides of equation (18) by $\text{Sin}\,\frac{m\pi x}{l}\,\delta x$ and integrating between $x = 0$ and $x = 2l$ or taking $2l$ times the average value of each term is to give us the equation

$$\frac{2l}{m\pi}=lA_m$$

or $$A_m=\frac{2}{m\pi}.$$

Hence for the expansion of $\frac{l-x}{l}$ we have

$$\frac{l-x}{l}=\frac{2}{\pi}\left\{\operatorname{Sin}\frac{\pi x}{l}+\frac{1}{2}\operatorname{Sin}\frac{2\pi x}{l}+\frac{1}{3}\operatorname{Sin}\frac{3\pi x}{l}+\text{etc.}+\frac{1}{m}\operatorname{Sin}\frac{m\pi x}{l}\right\} . \quad (20)$$

Therefore the potential at any point x in the cable at zero time or when $t = 0$ is expressed by

$$v_0=V\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{1}{m}\operatorname{Sin}\frac{m\pi x}{l}\right) \quad . \quad . \quad . \quad (21)$$

where Σ stands for the sum of a number of terms like $\frac{1}{m}\operatorname{Sin}\frac{m\pi x}{l}$, m being given various values, from $m = 1$ to $m =$ infinity.

Each of these terms is therefore a term of the type $A \operatorname{Sin}\frac{m\pi x}{l}$. We can therefore find an expression for the potential at any point in the cable after the lapse of a time t when the initial distribution is left to subside by simply multiplying each sine term of the above series by a factor of the type ϵ^{-m^2ut}, as already explained.

If then we denote by v_0 the potential at a distance x at a time $t = 0$, and by v_t the potential at x after a time t, we can express v_0 and v_t as follows:

$$v_0=V\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{1}{m}\operatorname{Sin}\frac{m\pi x}{l}\right) \quad . \quad . \quad . \quad (22)$$

$$v_t=V\frac{2}{\pi}\sum_{m=1}^{m=\infty}\left(\frac{1}{m}\epsilon^{-m^2ut}\operatorname{Sin}\frac{m\pi x}{l}\right) \quad . \quad . \quad . \quad (23)$$

Suppose next that we alter the origin of time, and, instead of reckoning the origin of time from the instant when the sending end is earthed after having been raised to a potential V and kept there long enough for the whole potential distribution to reach a steady state, let us suppose that the sending end has a battery applied to it or a source of steady potential V, and that we reckon the time from this instant of applying the voltage V to the sending end. At that instant when $t = 0$, the potential at the sending end jumps up to V, and at all other points rises up gradually to a limit which is given by the expression (22).

Hence at any time t reckoned from the instant of applying the steady voltage to the sending end, the potential v at any distance x from that sending end is given by the difference between the values of v_0 and v_t, as given in (22) and (23). In other words, if we apply a steady potential V to the sending end at a time $t = 0$, then at a time t and at a distance x the potential in the cable is given by

$$v=V\left[\frac{2}{\pi}\left\{\sum_{m=1}^{m=\infty}\left(\frac{1}{m}\operatorname{Sin}\frac{m\pi x}{l}\right)-\sum_{m=1}^{m=\infty}\left(\frac{1}{m}\epsilon^{-m^2ut}\operatorname{Sin}\frac{m\pi x}{l}\right)\right\}\right] \quad (24)$$

The part of the expression in square brackets will be denoted by $\phi\,(x,\,t)$, so that

$$v=V\phi\,(x,\,t) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

gives the potential at any time and place. This function $\phi\,(x,\,t)$ satisfies all the conditions. It satisfies the differential equation $\frac{d^2v}{dx^2} = RC\,\frac{dv}{dt}$, for it is the difference of two expressions which separately satisfy it. It also fulfils the boundary con-

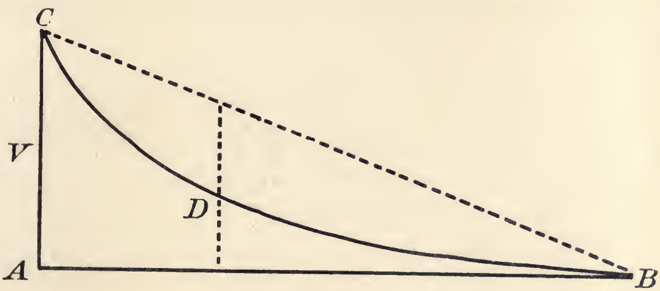

FIG. 2.

ditions, because when $t = 0$ $\phi\,(x,\,t) = 0$, and when $t =$ infinity $\phi\,(x, t) = \frac{l-x}{l}$. Hence it must be the expression for the potential in the cable at a distance x and at a time t.

We may represent it graphically as follows :—Let AB (Fig. 2) represent the cable, A being the sending end. Let a voltage V be applied at the sending end, represented by AC. Then at a time t, after the application of this voltage, the potential all along the cable will be represented by the ordinates of the firm line curve CDB. After a long time this potential everywhere approximates to a uniform fall represented by the ordinates of the dotted line CB. The ordinate of the firm line curve corresponding to any distance x represents the potential v and is given by the expression $v = V\phi(x, t)$. The current i in the cable at any point is obtained from the potential v by differentiation with regard to x, since by Ohm's law

$$i = -\frac{1}{R}\frac{dv}{dx} \quad . \quad . \quad . \quad . \quad (26)$$

Hence, performing the operation denoted by (26) on $v = V\phi(x, t)$, we obtain the expression for the current i at any time t and any distance x, viz.,

$$i = \frac{V}{R}\left\{\frac{1}{l} + \frac{2}{l}\sum_{m=1}^{m=\infty} \epsilon^{-m^2ut} \operatorname{Cos} \frac{m\pi x}{l}\right\}. \quad . \quad . \quad (27)$$

The current at the receiving end will be denoted by I_r, and it is obtained from (27) by putting $x = l$ and giving m increasing integer values from 1 to ∞. Hence

$$I_r = \frac{V}{Rl} 2\left\{\frac{1}{2} - \epsilon^{-ut} + \epsilon^{-4ut} - \epsilon^{-9ut} + \text{etc.}\right\} \quad . \quad . \quad (28)$$

It is convenient to denote ϵ^{-ut} by θ and to write (28) in the form

$$I_r = \frac{V}{Rl} 2\left\{\frac{1}{2} - \theta + \theta^4 - \theta^9 + \theta^{16} - \theta^{25} + \theta^{36} - \text{etc.}\right\} \quad . \quad (29)$$

The above is the expression for the current flowing into the earth at the receiving end at any time t after applying a steady voltage V at the sending end. Since θ is a proper fraction, the series in the brackets in (29) is rapidly convergent, and in general it is quite sufficient to take the sum of the first six or seven terms to obtain a close approximation to the actual value.

If we are given the numerical value of the whole resistance of the cable in ohms, which is equal to Rl, where l is the length, and the whole capacity of the cable in farads, which is equal to Cl, then we can at once calculate $u = \frac{\pi^2}{CRl^2} = \frac{9{\cdot}87}{Cl{\cdot}Rl}$, and hence we can calculate $\epsilon^{-ut} = \theta$ from the expression

$$\theta = \epsilon^{-ut} = \operatorname{Cosh} ut - \operatorname{Sinh} ut$$

for any assigned value of the time t. We can then find θ^4, θ^9, etc., easily by the use of a slide rule or table of logarithms. For $\log_{10}\theta^4 = 4 \log_{10}\theta$, and therefore $\theta^4 = \log_{10}{}^{-1}(4 \log_{10} \theta)$, etc. It is most convenient to arrange the series as follows:

$$\frac{1}{2} + (\theta^4 + \theta^{16} + \theta^{36} + \text{etc.}) - (\theta + \theta^9 + \theta^{25} + \text{etc.})$$

We shall denote the above series by $f(u, t)$. Accordingly we have for the received current

$$I_r = \frac{2V}{Rl} f(u, t) \quad . \quad . \quad . \quad . \quad (30)$$

and for any assigned value of the time t we can calculate the current I_r flowing to earth at the receiving end.

4. Curves of Arrival.—The series denoted by $f(u, t)$ has the curious property that its value is zero for all values of t from $t = 0$ up to $t = CRl^2 \times 0{\cdot}0233$ nearly.

Consider the series

$$\theta - \theta^4 + \theta^9 - \theta^{16} + \theta^{25} - \theta^{36}, \text{ etc.}$$

Assume $t = 0$; then $\theta = \epsilon^{-ut} = 1$, and the series (28) becomes equal to $1 - 1 + 1 - 1 + 1 - 1 + 1$, etc., to infinity. Let the sum of this last series to infinity be denoted by S; then

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1, \text{ etc.}$$

Hence $$S - 1 = -1 + 1 - 1 + 1 - 1 + 1 - 1, \text{ etc.}$$

Adding the above two series, we have

$$2S - 1 = 0 \text{ or } S = \frac{1}{2}.$$

Accordingly the sum $1 - 1 + 1 - 1 + 1$, etc., to infinity is equal to $\frac{1}{2}$, and therefore the series

$$f(u, t) = \frac{1}{2} - \theta + \theta^4 - \theta^9 + \theta^{16} - \theta^{25} + \theta^{36}, \text{ etc.},$$

is equal to zero when $\theta = 1$.

Also it can be shown by trial that for any value of θ between $\theta = 1$ and $\theta = 0{\cdot}8$ or $0{\cdot}9$ the value of $f(u, t)$ is zero.

Thus if $\theta = 0{\cdot}79$ we can easily find that $\theta^4 = 0{\cdot}389$, $\theta^9 = 0{\cdot}119$, $\theta^{16} = 0{\cdot}023$, and $\theta^{25} = 0{\cdot}003$.

Hence $\theta + \theta^9 + \theta^{25} = 0{\cdot}912$ and $\theta^4 + \theta^{16} = 0{\cdot}412$. Therefore

$$\frac{1}{2} + (\theta^4 + \theta^{16}) - (\theta + \theta^9 + \theta^{25}) = 0,$$

and $f(u, t) = 0$ when $\theta = \epsilon^{-ut} = 0{\cdot}79$. Also it can be shown that if $\theta = 0{\cdot}9$, then $\theta + \theta^9 + \theta^{25} = 1{\cdot}38$, and $\theta^4 + \theta^{16} = {\cdot}88$, and therefore $f(u, t) = 0$.

Lord Kelvin originally gave $\theta = 0{\cdot}75$ as the limiting value required to make $f(u, t)$ equal to zero, and he denoted the time corresponding to this by the letter a.[1]

Since $\theta = \epsilon^{-ut}$, we have $t = \frac{1}{u} \log_\epsilon \left(\frac{1}{\theta}\right)$, and if $\theta = 0{\cdot}75$ then $t = \frac{1}{u} \log_\epsilon \left(\frac{4}{3}\right)$. Hence Lord Kelvin's symbol a is a time such that

$$a = \frac{CRl^2}{\pi^2} \log_\epsilon \left(\frac{4}{3}\right).$$

Professor Fleeming Jenkin, another great telegraphic authority, gave as the limiting value $\theta = 0{\cdot}79 = 10^{-0{\cdot}1}$.

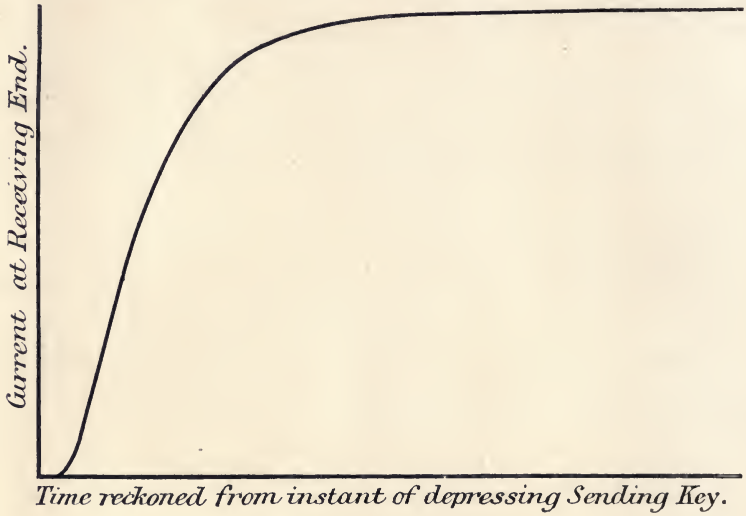

FIG. 3.—Curve of Arrival.

Now $\log_\epsilon (10^{0{\cdot}1}) = 0{\cdot}23$, and $\pi^2 = 9{\cdot}87$.

Accordingly we can say that

$$a = \frac{Cl.Rl}{9{\cdot}87} \times 0{\cdot}23 = \mathbf{CR} \times 0{\cdot}0233 \quad . \quad . \quad . \quad (31)$$

where **C** and **R** denote the capacity in farads and resistance in ohms of the whole cable.

Hence if the key is put down at the sending end connecting that end with a battery of constant potential V, then during an

[1] See Lord Kelvin, "On the Theory of the Electric Telegraph," *Proc. Roy. Soc., London*, May, 1855, or "Mathematical and Physical Papers," Vol. II., p. 71.

interval of time equal to a defined as above, no current capable of being detected by any receiving instrument, however sensitive, would be found flowing to earth at the receiving end. If, however, the sending key is kept down, then the current will begin to rise at the receiving end and steadily increase. After an interval equal to about $4a$ it will reach nearly half its final value, and after an interval $10a$ it will reach a final steady value.

If we plot a curve the ordinates of which denote to some scale the received current and the abscissæ the time reckoned

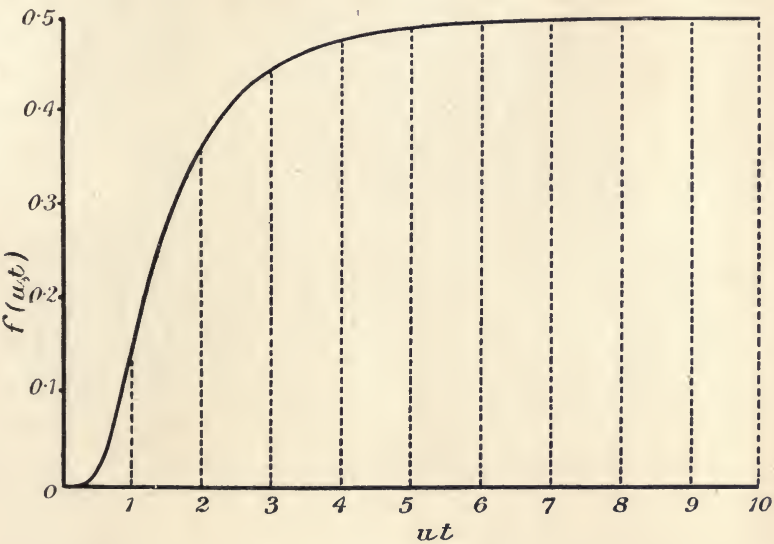

Fig. 4.—Curve of Arrival.

from the instant of applying the battery at the sending end, the curve so drawn is called a *curve of arrival.* It is generally drawn with abscissæ representing ut and ordinates representing $f(u, t)$, and has the form represented in Fig. 3.

Lord Kelvin was the first to give in 1855 curves of arrival drawn for different conditions.

The table below gives values of $f(u, t)$ for various values of ut calculated by Professor J. D. Everett, and the curve in Fig. 4 graphically represents these values.

The value of $f(u, t)$ approximates to 0·5 as ut reaches a value of about 10 and upwards. Below $u = 0·23$ $f(u, t) = 0$.

ut.	$f(u, t)$.	ut.	$f(u, t)$.	ut.	$f(u, t)$.
0·1	·000	1·5	·279	2·9	·445
0·2	·000	1·6	·300	3·0	·450
0·3	·001	1·7	·318	3·1	·455
0·4	·006	1·8	·335	3·2	·459
0·5	·018	1·9	·350	3·3	·463
0·6	·037	2·0	·365	3·4	·467
0·7	·062	2·1	·378	3·5	·470
0·8	·091	2·2	·389	3·6	·473
0·9	·121	2·3	·400	3·7	·475
1·0	·150	2·4	·409	3·8	·478
1·1	·179	2·5	·418	3·9	·480
1·2	·207	2·6	·426	4·0	·482
1·3	·233	2·7	·433	5	·493
1·4	·257	2·8	·439	10	·500

The interval of time approximately equal to 0·0233 multiplied by the product of the total resistance of the cable in ohms and its total capacity in farads is called the "silent interval," and, no matter what the voltage applied at the sending end, no measurable current will flow out at the receiving end to earth until after the lapse of this time.

After a time about ten times the silent interval has elapsed the current at the receiving end will have reached its full possible value. The possible speed of signalling is therefore closely connected with the duration of the silent interval. Since the silent interval a varies inversely as the value of u for the cable and as u varies inversely as the product CRl^2 or the product of the total resistance and total capacity, we can say that cables have equal sending power for which the value of CRl^2 is the same.

For any given type of receiving instrument the apparent time occupied in the transmission of a signal varies as the square of the length of the cable for cables of equal capacity and resistance per unit of length. The curve of arrival can be actually drawn by such a receiving instrument as the syphon recorder.

5. The Transmission of Telegraphic Signals along a Cable.—We have next to consider the mode of making, and the effect of transmission along the cable on telegraphic signals.

The alphabetic code usually employed in cable telegraphy is the International Morse Alphabet, according to which each

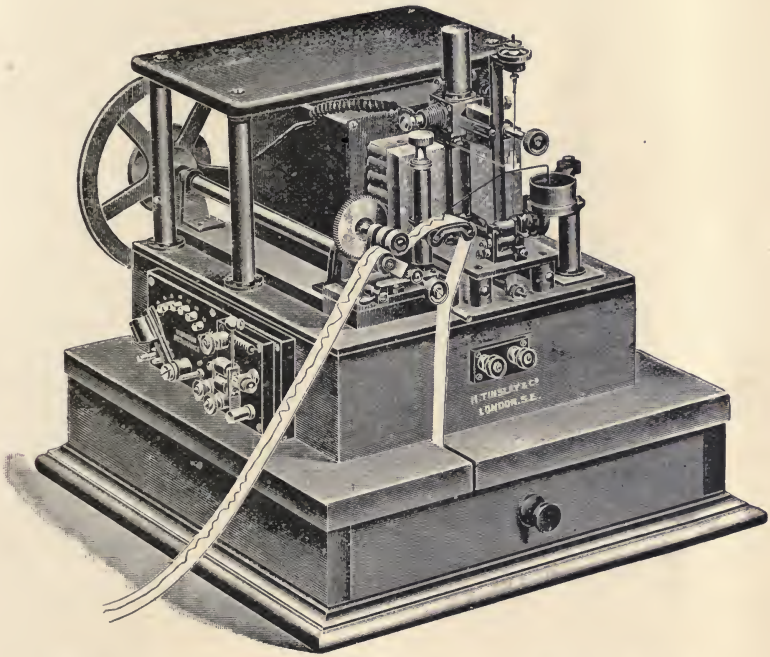

Fig. 5.—Syphon Recorder for Submarine Cable working as made by H. Tinsley & Co.

letter of the alphabet is denoted by one or more intermittent applications of a constant potential battery to the sending end of the cable, such application being made by a key which connects the cable to the battery for a certain short interval of time.

The battery of voltaic cells used has its centre connected to the earth, and a key is employed which connects either one or other terminal of the battery to the sending end of the cable and therefore raises it either to a positive potential $+ V$ or lowers it to a negative potential $- V$.

In signalling over land lines by hand-made signals the alphabetic signals are composed of short and long signals called respectively a *dot* and a *dash*.

Thus the letter A is represented by a *dot* followed by a *dash*

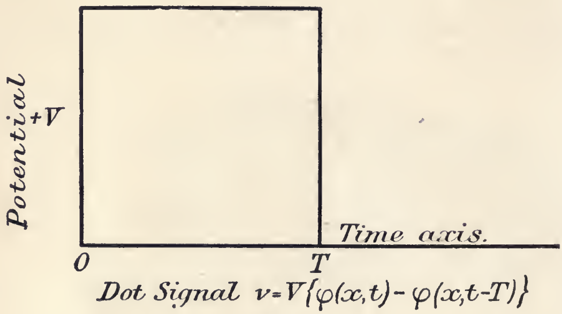

FIG. 6.

(– ——). The *dot* is made by connecting the sending end of the line for a short interval of time with one terminal of a battery. This is then removed and after an equal space of time connected again for a period about three times as long to form the *dash*.

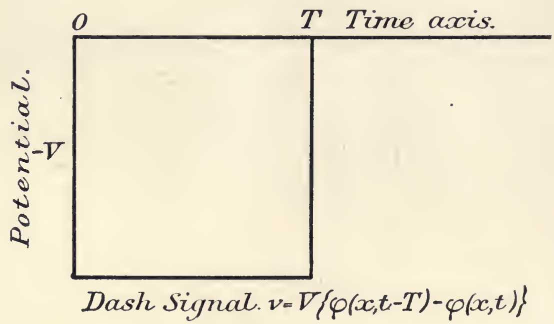

FIG. 7.

The currents into line are thus always in the same direction, but vary in duration.

In the case of cable signalling the currents which form the *dot* and *dash* signals are always of the same duration, but differ in sign or direction, those forming the dashes being say positive currents and those forming the dots being negative currents. The receiving instruments are therefore differently constructed.

For the land line hand sending either a needle instrument or else a Morse Inker is employed when printed signals are required, and the message is printed down in *dots* and *dashes* on paper strip.

In the case of submarine cables the receiving instrument used is the syphon recorder in which a delicate pen moves over a strip of paper, and the *dot* and *dash* signals are made by slight but sudden deflections to the right or left (see Fig. 5).

To make a dot signal the positive battery pole is applied to the sending end of the cable and causes the potential there to rise suddenly to $+ V$. After an interval of time T the battery is removed and the end put to earth. The variation of potential at the sending end may therefore be represented by the line in Fig. 6.

To make a dash signal the same process is followed with the reversal of the battery pole, so that the variation of potential at the sending end in making the dash signal is represented by the firm line in Fig. 7.

We have then to consider the nature of the potential changes at distant points in the cable and of the current flowing out at the receiving end.

We may regard the dot signal as created by applying to the sending end a source of positive potential and keeping it on for an infinite time, but after the lapse of a time T superimposing upon that state the application of an equal source of negative potential which reduces the sending end to zero and keeps it zero.

We have seen that the effect at distant points in the cable of applying a potential $+ V$ at the sending end is to raise the potential at a point at a distance x after a time t to a value $v = V \phi(x, t)$. Hence the effect of applying a negative potential $- V$ after the lapse of the time T is represented by $v = - V \phi(x, (t - T))$. Hence the potential in the cable at any distance x due to a *dot* signal made at the sending end is represented by

$$v = V \{\phi(x, t) - \phi(x, t - T)\} \quad . \quad . \quad . \quad (32)$$

Also the potential due to a *dash* signal is represented by

$$v = V \{\phi(x, (t - T)) - \phi(x, t)\} \quad . \quad . \quad . \quad (33)$$

Again, we have seen that the effect of applying a source of potential $+V$ to the cable at the sending end and keeping it on is to cause a current i to flow out at the receiving end which is represented by
$$i=\frac{2V}{Rl}f(u, t).$$

Hence the effect of making a *dot* signal at the sending end must be to cause a current at the receiving end represented by
$$i=\frac{2V}{Rl}\{f(u, t)-f(u, (t-T))\} \quad . \quad . \quad . \quad (34)$$
and similarly the effect of making a *dash* signal at the sending end must be to cause a current at the receiving end represented by
$$i=\frac{2V}{Rl}\{f(u, (t-T))-f(u, t)\} . \quad . \quad . \quad (35)$$

We can therefore select any combination of dot and dash signals, in other words any letter of the alphabet, and predict exactly by an equation the current which will at any instant be found at the receiving end of the cable flowing into or out of the earth. The expressions (34) and (35) are in fact the equations to the curves representing the dot and dash signals as recorded at the receiving end by a syphon recorder or some equivalent instrument.

Thus, for instance, let us consider the nature of the received current corresponding to a dot signal.

We may consider the constant factor $2V/Rl$ to be unity and the duration T of the dot such that $uT=\frac{\pi^2}{CRl^2}T$ is, for example, 0·3. Then we have $\theta=\epsilon^{-ut}$ and $\theta_1=\epsilon^{-u(t-T)}=\epsilon^{-ut}\times\epsilon^{uT}=k\theta$, say. Then $f\ (u,\ t)=\frac{1}{2}-\theta+\theta^4-\theta^9+\theta^{16}-\theta^{25}$, etc., and $f(u, (t-T))=\frac{1}{2}-\theta_1+\theta_1{}^4-\theta_1{}^9+\theta_1{}^{16}-\theta_1{}^{25}$, etc.

If we assign to ut various increasing values, 0·4, 0·5, 0·6, etc., we can calculate the values of
$$\theta=\epsilon^{-ut}=\text{Cosh } ut-\text{Sinh } ut,$$
$$\theta^4=\epsilon^{-4ut}=\text{Cosh } 4ut-\text{Sinh } 4ut,$$
$$\theta^9=\epsilon^{-9ut}=\text{Cosh } 9ut-\text{Sinh } 9ut,$$
and so on, and hence obtain the value of $f(u, t)$ in the form

$$f(u, t) = \frac{1}{2} - \text{Cosh } ut + \text{Sinh } ut + \text{Cosh } 4ut - \text{Sinh } 4ut$$

$$- \text{Cosh } 9ut + \text{Sinh } 9ut + \text{Cosh } 16ut - \text{Sinh } 16ut - \text{etc.} \quad . \quad (36)$$

These values are easily obtained from any good table of hyperbolic functions. We then find the value of ϵ^{uT} from the equation $k = \epsilon^{uT} = \text{Cosh } uT - \text{Sinh } uT$.

Hence $\qquad \theta_1 = k\,(\text{Cosh } ut - \text{Sinh } ut),$

$\qquad\qquad \theta_1{}^4 = k^4\,(\text{Cosh } 4ut - \text{Sinh } 4ut)$, etc.

Therefore

$$f(u, (t-T)) = \frac{1}{2} - k \text{ Cosh } ut + k \text{ Sinh } ut + k^4 \text{ Cosh } 4ut - k^4 \text{ Sinh } 4ut$$

$$- k^9 \text{ Cosh } 9ut + k^9 \text{ Sinh } 9ut, \text{ etc.} \quad . \quad . \quad . \quad (37)$$

This series can be calculated without difficulty by means of a table of hyperbolic functions and one of powers of ϵ.

It is then easy to find, by subtracting the sums of the two series (36) and (37), the value of $f(u, t) - f(u, (t-T)) = f(ut, T)$ for various values of ut.

Thus, if $uT = 0{\cdot}3$, the following values of the above function were calculated by Everett:

ut.	$f(ut) - f(ut - 0{\cdot}3)$.	ut.	$fut - f(ut - 0{\cdot}3)$.
0·4	6	2·3	35
0·5	18	2·4	31
0·6	36	2·5	29
0·7	56	2·6	26
0·8	73	2·7	24
0·9	84	2·8	21
1·0	88	2·9	19
1·1	88	3·0	17
1·2	86	3·1	16
1·3	83	3·2	14
1·4	78	3·3	13
1·5	72	3·4	12
1·6	67	3·5	11
1·7	61	3·6	10
1·8	56	3·7	8
1·9	50	3·8	8
2·0	47	3·9	7
2·1	43	4·0	7
2·2	39		

The curve representing the above values or the "curve of arrival" for this *dot* signal is shown plotted in Fig. 8. It will be seen, therefore, that the effect of pressing down the sending

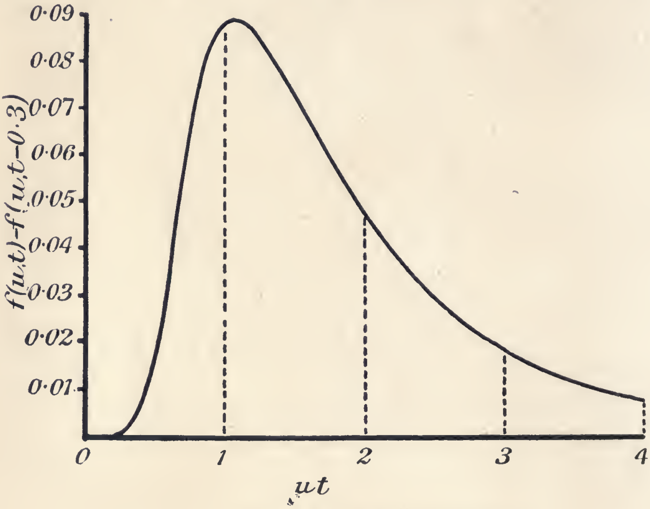

FIG. 8.—Curve of Arrival of Dot Signal.

key for a short time and applying a brief constant steady voltage to the sending end appears at the receiving end in the form of a current which rises up gradually to a maximum value and then fades away. Hence these dot signals cannot be repeated

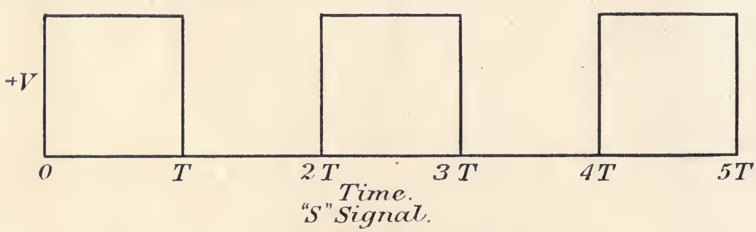

FIG. 9.—"S" Signal as sent.

faster than a certain limiting speed, or else the effect at the receiving end is indistinguishable from a prolonged dash signal. We here see the reasons for the limitation of the speed of cable telegraphy. The larger the value of CRl^2 or of the product **CR**, viz., the product of the total capacity in farads and resistance in ohms of the cable, the smaller the value of u, and the longer will be the time before the current at the receiving end reaches its maximum value after the sending key is depressed. Also, the smaller the value of u, the less will be the maximum value of the received current, and in general the less quickly can the intermittent signals succeed each other consistently with retaining an interpretable form at the receiving end.

The above method of calculation enables us to predict the form of the curve representing the received current as a function of the time for any assigned signal made with the key at the sending end. Thus, for instance, take the letter S. This is

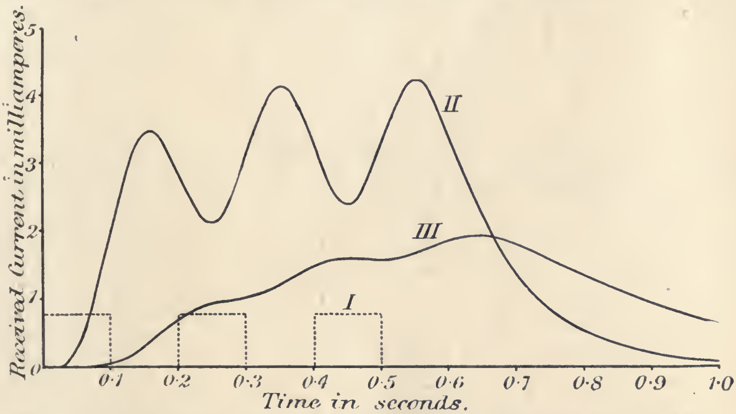

FIG. 10.—The dotted line represents the " S " Signal as sent, and the firm lines as received on Cables of various CR values, and lengths. For Curve II. length = 1,000 miles, $CR = 1{\cdot}0$, and for Curve III., length = 1,581 miles, $CR = 2{\cdot}5$.

represented in the International Morse Alphabet by three dots, each space between the dot signals being equal in duration to that of the dot. Hence to make this signal the key at the sending end is tapped three times, and this applies to the sending end of the cable a variation of potential V, represented by the curve in Fig. 9.

Let the duration of each dot and each space be represented by T. Then the current at the receiving end is expressed as a function of the time by the equation

$$I_r = \frac{2V}{Rl}\Big\{f(ut) - f(u(t-T)) + f(u(t-2T)) - f(u(t-3T)) + f(u(t-4T)) - f(u(t-5T))\Big\} \quad . \quad (38)$$

To calculate I_r we have to give to the symbol t various increasing values, 0·1, 0·2, 0·3, etc., and calculate the value of the function on the right-hand side of the expression (38). To do this we must have the length of the cable l, the sending voltage V, and the capacity C and resistance R per mile given. We can then calculate $\frac{2V}{Rl}$ and $u = \frac{\pi^2}{CRl^2}$. Also the value of T must be given in fractions of a second, so that uT is known.

With some considerable labour the value of I_r for various values of t can be calculated and the curve of arrival for the S signal graphically depicted. This has been done for the author by Mr. G. B. Dyke as shown in Fig. 10, which represents the form of the curve of arrival for an S signal on certain hypothetical cables.

6. The Speed of Signalling: Comparison of Different Cables.—Every type of receiving instrument used for recording telegraphic signals is characterised by requiring a certain minimum current to actuate it. Hence, in order that the particular instrument used may record a legible signal, it must be traversed by a current of not less than this critical value and for a certain period of time. We have seen that the current at the receiving end of the cable is a function of the quantity ut. For the same value of ut and for the same mode of working or making the signal the current at the receiving end will be the same.

It is therefore necessary to have a particular minimum value of ut below which no signal will be recorded. Accordingly this value of ut may be taken as a working constant. Now the cable has a particular value of $u = \frac{\pi^2}{CRl^2}$, which is characteristic of it, and hence the time required to establish the minimum or necessary working current at the receiving end for a given cable and impressed voltage varies inversely as u or directly as CRl^2. Hence for cables made in the same manner, but of various lengths, this time varies as the square of the length. The speed of signalling varies inversely as the time required for the received current to reach the minimum strength, as it is clear the signals cannot succeed each other more frequently than N per second where $1/N$ is the time required to affect the receiving instrument. Hence the signalling speed varies inversely as the product CRl^2 and inversely as the square of the length for cables of the same make.

This means that there is no definite "velocity of electricity." The interval of time which elapses between closing the circuit at the sending end and recording the signal depends not only on the sending voltage, but upon the nature of the receiving instrument and upon the length of the cable. This explains how it is that the older electricians and telegraphists obtained such very various and different results in their endeavours to measure the supposed velocity of electricity along a wire or cable.

The speed of signalling can be increased by decreasing the total resistance and total capacity of the cable. This latter, however, is not much under control, as it is determined chiefly by the dielectric constant of the insulator which is used, and for submarine cables no substance has yet been found to take the place of gutta-percha. Accordingly the increase in speed chiefly depends upon an increase in the diameter of the copper conductor. Long cables must therefore necessarily be heavy cables if we are to preserve reasonable speed in signalling. An empirical rule for speed of signalling is given in Mr. Jacobs' article "Submarine Telegraphy" in the *Encyclopædia Britannica* (supplement to the tenth edition) as follows: If S is the number of five-letter words which can be sent per minute through a cable when using the Kelvin syphon recorder as receiver, and if $\mathbf{C}$ is the total capacity and $\mathbf{R}$ the total resistance of the cable, then $S = \frac{120}{\mathbf{CR}}$. The capacity must be measured in farads and the resistance in ohms.

For example, suppose a cable 3,142 nautical miles or nauts in length to have a resistance of three ohms per naut and a capacity of 0·33 microfarad per naut. Then

$$\mathbf{CR} = \frac{0{\cdot}33 \times 3}{10^6} \times (3{,}142)^2 = 9{\cdot}87,$$

and $u = \frac{\pi^2}{CRl^2} = 1$, since $\pi^2 = 9{\cdot}87$ nearly.

Hence by the above rule $S = \frac{120}{9\cdot87} = 12 - 13$, and the sending speed would be twelve to thirteen five-letter words, or sixty to sixty-five letters per minute.

We are therefore able to predict not only the form of the current curve at the receiving end for a given kind of signal made at the sending end, but also the speed with which the signals can succeed each other in cables with various values of C, R, and l.

7. Curb-sending.—It will be clear from the above explanations that the obstacle to signalling speed is the effect

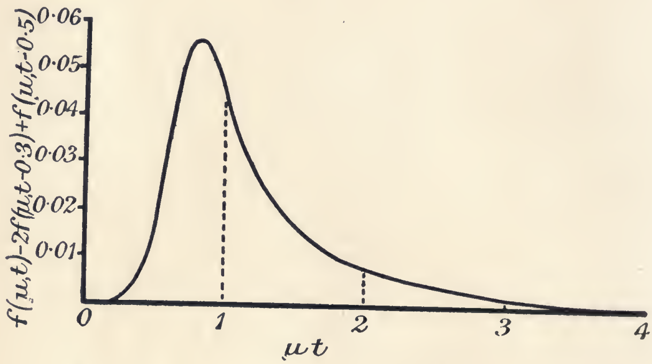

Fig. 11.—Curve of Arrival for Curbed Dot Signal.

of the capacity and resistance of the cable in dragging out a sharply made signal or voltage change made at the sending end into a slow rise and fall of current at the receiving end. Hence until the cable is cleared of a previous signal another one cannot be usefully despatched, or if it is the two run together into a received signal indistinguishable as two.

One method by which speed of signalling can be increased is by means of *curb-sending*.

By this method in sending a dot signal the cable at the sending end is first raised a positive potential for a certain time, then lowered instantly to an equal negative potential, and after about two-thirds of the above time put again to earth. In other words, we send into the cable a current in one direction and then follow it instantly by another in the opposite direction for a somewhat shorter time. The effect of this is to clear the cable more quickly for the following signal.

The operation at the sending end may be represented by a rectangular line, which shows the application of a positive potential to the cable followed by an equal negative potential for a shorter time, and then by an earthing or reduction to zero potential.

Let us consider then the effect of the above operation carried out at the sending end upon the cable at other different points.

If $+V$ and $-V$ are the positive and negative potentials applied to the sending end, the former for a time T_1 and the latter for a time $T_2 - T_1$, then the potential v at any distance x along the cable at any time t is given by

$$v = V\{\phi(x_1 t) - 2\phi(x_1(t - T_1)) + \phi(x(t - T_2))\}$$

and the received current by

$$I_r = \frac{2V}{Rl}\left\{f(ut) - 2f(ut - uT_1) + f(ut - uT_2)\right\}.$$

Thus, for instance, if the value of $uT_1 = 0{\cdot}3$ and $uT_2 = 0{\cdot}5$, then the values of the received current have been calculated by Professor Everett on the assumption that the factor $2\,V/Rl = 1$ for various values of ut as follows:

ut.	$f(ut) - 2f(ut - 0{\cdot}3) + f(ut - 0{\cdot}5)$.	ut.	$f(ut) - 2f(ut - 0{\cdot}3) + f(ut - 0{\cdot}5)$.
0·4	6	1·5	15
0·5	18	1·6	13
0·6	35	1·7	11
0·7	50	1·8	10
0·8	56	1·9	9
0·9	53	2·0	8
1·0	44	2·1	8
1·1	34	2·2	7
1·2	27	2·3	5
1·3	24	2·4	5
1·4	20	2·5	5

If these values are plotted out we obtain a curve of the form shown in Fig. 11.

On comparing it with the curve in Fig. 8 representing the uncurbed signal it is seen that the uncurbed signal rises more slowly and dies away more slowly, but it has a larger maximum value than the curbed signal.

It is found that if condensers are inserted in series with the cable both at the sending and receiving end the effect is to curb the signals to a considerable extent. In modern practice the cable, however, is nearly always *duplexed*, that is to say arranged with an *artificial line* of equal total capacity and resistance in the manner shown in Fig. 12.

In this case C_1 and C_2 are two large condensers. C is the cable, and C_3 is an artificial line which consists of sheets of tinfoil placed on one side of sheets of paraffined paper, the

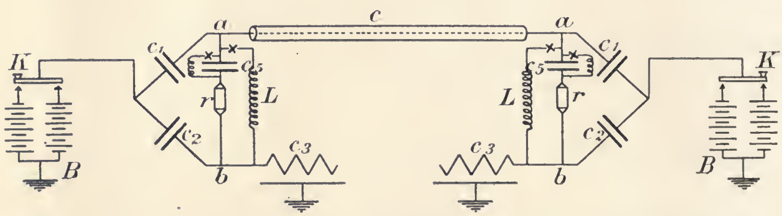

Fig. 12.—Arrangements for Duplex Transmission in a Submarine Cable.

opposite side of the paper sheet being coated with a strip of tinfoil cut in zigzag fashion. The zigzag tinfoil strip has resistance and capacity with respect to the other sheet of metal, which is earthed. Such a line can be adjusted to represent a cable of any length and of any capacity and resistance per unit of length. The receiving instrument, generally a syphon recorder r, is connected between the ends of the real and artificial cable, and another condenser C_5 is placed in series with it. The battery B and sending key K are joined in as shown. The artificial line can so be balanced against the real line that on depressing a key the current flows equally into the two condensers C_1 and C_2 and into the real and artificial lines, and the points a and b remain at the same potential. Hence the current sent out through the cable does not affect the local receiving instrument.

On the other hand, if a current arrives it flows to earth partly

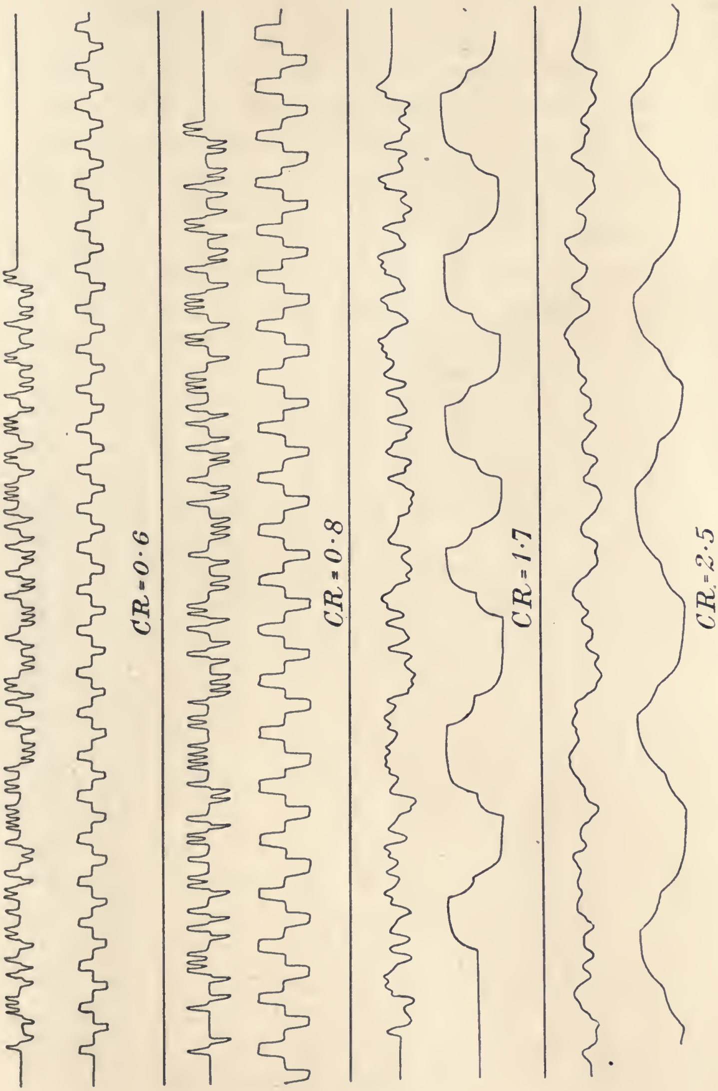

FIG. 13.—The above are Syphon Recorder Signals taken by Mr. H. Tinsley as received on Cables having values of *CR* as given below each line. The upper line is an alphabet and the lower line a succession of *dot dash* to show the rounding effect on the signals of gradually increasing the value of *CR*.

through the receiving instrument and the artificial line and partly to earth through the local battery. The cable is then duplexed, and signals can be sent and received at the same moment.

It is now usual to dispense with the condenser C_5 in series with the recording instrument and in place of it to insert an inductive shunt L across the terminals of the coil of the syphon recorder. The effect of this inductive shunt is to curb the signals and clear the cable quickly for the next signal. The sudden quick rise of potential at the terminals of the recorder which accompanies the reception of the first part of the signal affects the recorder, but the slow fall which takes place after the maximum is past causes a current to flow through the inductive shunt, and the recorder coil falls back quickly to zero.

In the case of a short cable or one with small **CR** the signals made by the syphon recorder are sharp and well defined. The syphon recorder consists of a light coil of insulated wire hung by a bifilar suspension in the field of a strong magnet like a movable coil galvanometer. To this coil is attached a light glass pen, the point of which rests on a strip of paper tape which is moved by clockwork beneath the pen. If then the coil is at rest the pen traces a straight line along the centre of the tape. If a brief current from the cable is sent through the coil the latter is jerked on one side, and when the current ceases it falls back to its normal position.

The effect is to make a *dot* signal which is a square notch on the line if the cable is very short. If, however, the current rises up slowly and falls again slowly, then the ink line is a rounded mark. The *dash* is made by reversing the direction of the current and therefore of the motion of the pen. In the case of short cables the alphabetic signals made by groups of these dots and dashes are quite legible, but in the case of long cables it requires some skill to guess the meaning, since the marks on the tape are, as it were, parts of "curves of arrival" running into each other. The reproductions of syphon recorder tapes in Fig. 13 are from experiments kindly made for the author by Mr. H. Tinsley with artificial lines of different capacities and resistances to show this rounding effect on the signals with increasing values of **CR**.

CHAPTER VI

THE TRANSMISSION OF HIGH FREQUENCY AND VERY LOW FREQUENCY CURRENTS ALONG WIRES

1. The Modifications in the General Equation for Transmission in the Cases of very High and very Low Frequency.—Returning to the general equation for the transmission of electrical disturbances along a cable, we can write it in the form

$$\frac{d^2v}{dx^2}=CL\frac{d^2v}{dt^2}+(RC+SL)\frac{dv}{dt}+RSv \quad . \quad . \quad . \quad (1)$$

where v is the potential in the cable at a point at a distance x from the sending end and at a time t.

The above is the general equation for the propagation of potential changes of any type along a cable having resistance, capacity, inductance, and leakage. It may be called the *telephone equation.* It has been fully discussed in Chapter IV. Secondly, if the cable is such that L and S are very small relatively to R and C and if the frequency is low we can neglect the terms involving L and S and write the equation in the form

$$\frac{d^2v}{dx^2}=RC\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the case of the submarine telegraph cable, and the above equation (2) may therefore be called the *telegraph equation.* In this form it has been considered in Chapter V. Thirdly, if R and S are very small or negligible and if the frequency is very high we can neglect the terms involving R and S and write the equation (1) in the reduced form

$$\frac{d^2v}{dx^2}=CL\frac{d^2v}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since this applies in the case of electric oscillations or very high frequency alternating currents as employed in wireless telegraphy, we may call the above equation (3) the *radiotelegraphic equation.*

Lastly, if the line is an aerial line of small capacity and inductance operated at low frequency or with continuous current so that the principal constants are the resistance R and leakage S we can neglect L and C, and the general equation reduces to

$$\frac{d^2v}{dx^2}=RSv \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since this applies in the case of lines operated at very low frequency or with continuous currents and with such high voltage as to make the leakage important, we may call the above equation the *leaky line equation.*

Furthermore, if the variation of potential with time is simply harmonic, that is if the applied electromotive force is a simple sine curve *E.M.F.*, then, neglecting the effects at first contact, we can say that after a short time the variation of potential is simply harmonic everywhere and varies as the real part of ϵ^{jpt}. Hence $\frac{dv}{dt}=jpv$ and $\frac{d^2v}{dt^2}=-p^2v$. Accordingly the equations (1), (2), (3), and (4) above then take the form

$$\frac{d^2v}{dx^2}=\left\{-p^2CL+jp\,(RC+SL)+RS\right\}v \quad . \quad . \quad (5)$$

$$\frac{d^2v}{dx^2}=jpRCv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2v}{dx^2}=-p^2CLv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{d^2v}{dx^2}=RSv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have already discussed the equations (1) and (2) and (5) and (6) in Chapters IV. and V., dealing with telephony and submarine cable telegraphy. Hence we need not say more about them. The equations (3) and (7) and (4) and (8) remain, however, to be discussed.

2. The Propagation of High Frequency Currents along Wires.—Taking, then, the equation (3), viz.,

$$\frac{d^2v}{dx^2}=CL\frac{d^2v}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

we find that one particular solution applicable to the case considered is

$$v = \text{Cos}\, A\left(x + \frac{t}{\sqrt{CL}}\right) \quad . \quad . \quad . \quad . \quad (10)$$

For if we differentiate the above expression (10) twice with regard to x and twice with regard to t, we find that when the last expression is multiplied by CL it is the same as the former.

For $$\frac{d^2v}{dx^2} = -A^2\, \text{Cos}\, A\left(x + \frac{t}{\sqrt{CL}}\right)$$

and $$\frac{d^2v}{dt^2} = -\frac{A^2}{CL}\, \text{Cos}\, A\left(x + \frac{t}{\sqrt{CL}}\right).$$

Hence (10) is a solution of (9).

We see that it implies that v is periodic in space, that is, along the wire as well as with time. Therefore, in the case of a wire traversed by a high frequency current, at any one instant the potential varies along the line in a simple harmonic manner. If, however, we fix attention upon the variation of potential at any one point in the line, it is also periodic or varies as a simple cosine function of the time.

If we substitute $x + \frac{2\pi}{A}$ for x in the expression (10), whilst keeping t constant, we see that its value remains unaltered, because $\text{Cos}\,(\theta + 2\pi) = \text{Cos}\,\theta$. Hence at distances along the line equal to $\frac{2\pi}{A} = \lambda$ the potential value repeats itself. Accordingly this distance is the wave length of the potential along the line. If we keep x constant and substitute $t + \frac{2\pi\sqrt{CL}}{A}$ for t in (10) we see that its value also remains unchanged. Hence at any one point in the line the values of the potential repeat themselves at intervals of time equal to $T = \frac{2\pi\sqrt{CL}}{A}$. This is therefore the periodic time of the potential variation.

The velocity W with which the wave of potential travels is given by $W = \frac{\lambda}{T}$. Hence, since $\lambda = \frac{2\pi}{A}$ and $T = \frac{2\pi\sqrt{CL}}{A}$, we have

$$W = \frac{1}{\sqrt{CL}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If then we apply at the end of a very long wire having inductance L and capacity C per unit of length a simple periodic high frequency electromotive force, the effect will be to make waves of electric potential travel along the wire with a velocity $1/\sqrt{CL}$ centimetres per second, and at any one point in the line there will be oscillations of potential with a frequency $\frac{2\pi\sqrt{CL}}{A}$.

3. Stationary Oscillations on Finite Wires.—We are not much concerned practically with the propagation of high frequency currents along extremely long lines, but when the wires are of length less than or comparable with the wave length we may have the phenomena of stationary waves presented. Thus suppose a thin wire of not very great length, having a capacity C and inductance L per unit of length, to have a high frequency electromotive force applied in the centre, the frequency n being such that the quotient of $W = \frac{1}{\sqrt{CL}}$ by n, or $\frac{1}{n\sqrt{CL}}$ is equal to about twice the length of the wire. Then a wave of potential would run outwards in each direction and be reflected at the open ends of the wire and return again to find that the electromotive force had changed its phase by half a period. The oscillations of electromotive force are thus in step with the movements of the wave of potential, and therefore the latter are maintained and amplified. The whole process is exactly like that by which stationary oscillations are maintained on a rope fixed at one end by administering little jerks to the other end when held in the hand. The frequency of the jerks must agree with the interval of time taken by the wave motion to run along the rope and return.

Moreover, if we make jerks more quickly, say twice as quickly, the cord can accommodate itself to this increased frequency by dividing itself into two vibrating sections separated by a stationary point called a *node*, each *loop* or ventral segment being half the length of the cord.

In the same manner an experienced violinist, by lightly touching a string at one point and bowing at another, can cause the string to vibrate in sections and give out musical notes which are *harmonics* of the fundamental vibration. An exactly similar phenomenon can be exhibited electrically.

4. The Production of Loops and Nodes of Potential in a Conductor by High Frequency Electromotive Forces.—To obtain a conductor suitable for exhibiting these effects in a convenient space we require a conductor along which waves of electric potential travel rather slowly.

In the case of ordinary straight single wires of good conductivity, waves of electric potential travel along the wire with the speed of light, or about 1,000 million feet per second. If, therefore, we can create high frequency oscillations having a frequency of one million, the length of the wave of potential would be 1,000 feet or so, and we should require a wire 500 feet long to exhibit the phenomena. If, however, we coil a fine silk-covered wire on an ebonite rod so as to form a long helix of one layer of closely adjacent turns, we can make a conductor which will have a capacity of approximately the same value per unit of length as a metal cylinder of the same dimensions as the helix, but an inductance per unit of length much larger than that of any single wire.

If a long helix of insulated wire is made as above described such that the length is at least fifty times the diameter, the inductance per unit length of the helix will be $(\pi DN)^2$ absolute electromagnetic units of inductance, that is, centimetres, or $\frac{1}{10^9}(\pi DN)^2$ henrys, where D is the mean diameter of the helix and N the number of turns of wire per unit of length of the helix.

The capacity of such a helix will depend on its proximity to the ground, but if placed say 50 cms. above a table it will be given approximately by the expression $\dfrac{1{\cdot}5l}{2\log_e \dfrac{2l}{D}}$.

It will be found on trial that it is easy to construct a helix along which electric waves of potential will travel so slowly that for frequencies of one million or so the wave length will bear comparison with such lengths of helix as can be conveniently constructed.

Thus, for instance, on a round ebonite rod about $2\frac{1}{4}$ metres long the author wound a spiral of silk-covered No. 30 *S.W.G.* copper wire in a helix of one single layer 215 cms. long and having 5,470 turns. The helix had a mean diameter of 4·75 cms.

The inductance L of such a helix per unit of length is then given by

$$L = \left(\frac{3{\cdot}1415 \times 4{\cdot}75 \times 5470}{215}\right)^2 = 0{\cdot}149 \times 10^6 \text{ cms.}$$

The capacity per unit of length calculated by the formula $\dfrac{3}{4 \log_e \dfrac{2l}{D}}$ gave $C = 0{\cdot}187 \times 10^{-6}$ microfarads, and by actual measurement was found to be $0{\cdot}21 \times 10^{-6}$ microfarads when the helix was supported horizontally and 50 cms. above a table.

The velocity of propagation of a wave of electric potential along this helix is then equal to $1/\sqrt{CL}$, where $L = \dfrac{32}{215 \times 10^3}$ henry and $C = \dfrac{45}{215 \times 10^{12}}$ farad, and hence

$$W = \frac{1}{\sqrt{CL}} = \frac{215 \times \sqrt{1000} \times 10^6}{\sqrt{45 \times 32}} = 174 \times 10^6 \text{ cms. per second.}$$

The velocity of light is $30{,}000 \times 10^6$ cms. per second, and hence the velocity of a wave of potential along the above helix is only 1/172 part of that of the velocity of light.

If then we apply to the end of such a helix a high frequency alternating electromotive force having a frequency of about 200,000 per second, the result will be to create a wave of potential which travels a distance of four times the length of the helix in the time of one complete oscillation. For, the velocity of propagation being 174×10^6 cms. per second and the frequency 2×10^5, the corresponding wave length λ must be 870 cms., which is not far from four times 215.

An alternating *E.M.F.* of this frequency is best obtained by means of the oscillating discharge of a condenser.[1]

[1] For a full discussion of this mode of discharge the reader is referred to the following books by the Author: "The Principles of Electric Wave Telegraphy and Telephony," 2nd Edition, Chapter I. (Longmans & Co.); "An Elementary Manual of Radiotelegraphy and Radiotelephony," Chapter I. (Longmans & Co.).

If a condenser or Leyden jar of capacity C_1 is joined in series with an inductance L_1 and with a short spark gap, and if the spark balls are connected to an induction coil, oscillatory discharges of the condenser will take place through the inductance coil having a frequency given by the formula $n = \frac{1}{2\pi\sqrt{C_1 L_1}}$ where C_1 is measured in farads and L_1 in henrys, or else by the formula $n = \frac{5{\cdot}033 \times 10^6}{\sqrt{C_1 \times L_1}}$, where C_1 is measured in microfarads and L_1 in centimetres.

Thus the capacity of the condenser used was 0·005835 mfd. and the inductance of the coil was 110,000 cms. The frequency of the oscillations set up was therefore $0{\cdot}197 \times 10^6$, or nearly 200,000.

If the above-mentioned helix is connected to one end of the inductance coil and the other end of the coil is to earth, as shown in Fig. 1, then the oscillations set up in the inductance coil by the discharge of the condenser or Leyden jars create electric impulses on the end of the helix AB equivalent to the action of an electromotive force having a frequency of 197,000. The helix has thus produced upon it stationary waves of electric potential, and owing to the cumulative action the amplitude of the potential variation at different parts of the helix increases from a minimum at the end by which it makes contact with the condenser circuit to a maximum at the free end. At this last place the amplitude of potential variation may be so great that it reaches a value at which sparks and electric brushes fly off the end of the helix. In any case the gradual increase along the helix can be proved by holding near the helix a vacuum tube of the spectrum type (see Fig. 1) filled with the rare gas neon or in default one with carbon dioxide. The tube glows when held in a high frequency electric field, and the brilliancy of the glow will be found to decrease as the tube is moved from a place near the open end of the helix to a place near the end at which it is attached to the condenser circuit. We may represent this variation of potential along the helix by drawing a cylinder or double line to denote the helix and a dotted line in such position that the distance between the dotted line and the line representing the helix denotes the amplitude of the potential variation at that point in the helix.

An analogy is found in the case of a strip of steel held at one end in a vice and made to vibrate by pulling it on one side and letting it go. The amplitude of the motion of the different parts of the strip increases from zero at the bottom end, where it is gripped, up to a maximum at the free end. We can, however, make the above steel strip vibrate in such a manner that there is a node of vibration at a point about one-third of the way from the free end. In the same manner if we decrease the capacity

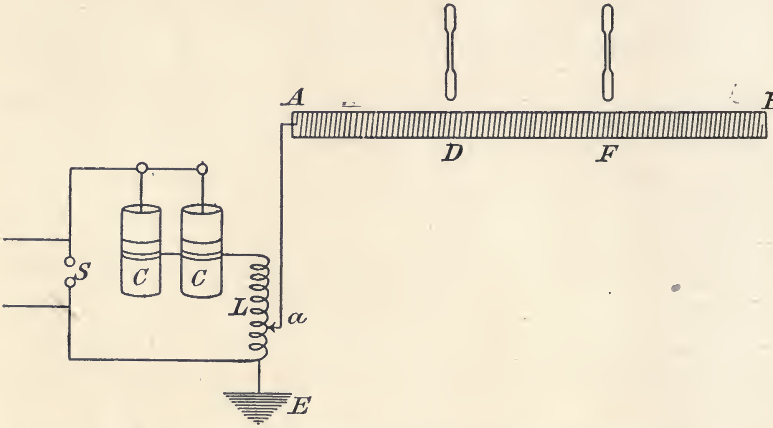

FIG. 1.—Arrangement of Apparatus for producing stationary electric oscillations on a helix *A B*. *C*, *C*, are Leyden Jars, *L* is an inductance coil, and *S* is a spark gap.

and inductance in the condenser circuit to which the helix is attached so as to make the frequency of the electromotive force acting on the end of the helix three times that required to produce the fundamental vibration, or say about 600,000 in the case of the helix above described, then the effect will be that to accommodate itself to the tripled frequency the stationary waves of potential on the helix must have a node of potential at about one-third of the way from the free end, and the distribution of potential amplitude can be denoted by the ordinates of the dotted line in Fig. 2.

In the same manner by increasing the frequency to 5, 7, 9, etc., times that required to excite the fundamental oscillations on the helix, we can create *harmonic* oscillations which have 2, 3, 4,

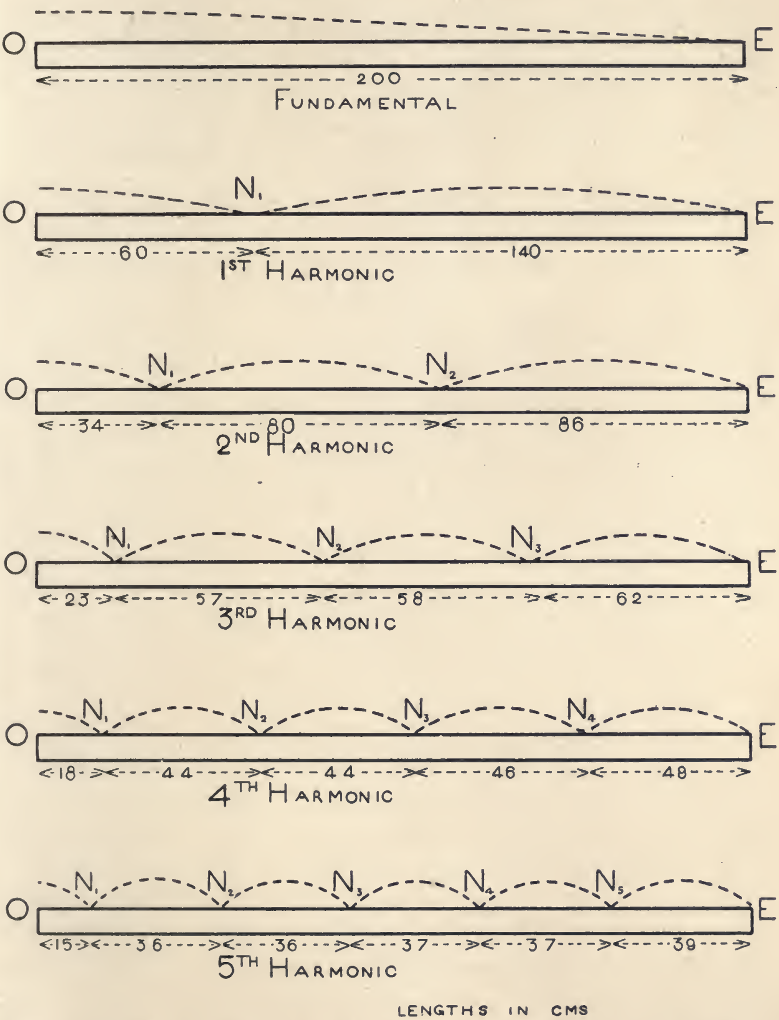

FIG. 2.—Diagram illustrating the formation of nodes and loops of potential upon a helix by means of electromotive forces of progressively increasing frequency.

etc., nodes of potential. The existence of these nodes can be proved by holding a neon vacuum tube near the helix and moving it along from one end to the other. When near a node the tube will not glow, but when opposite to an antinode or ventral segment it will glow very brightly.

The distance between two adjacent nodes is half a wave length of the stationary oscillations. Hence from this measured wave length λ and the calculated speed of propagation W we can determine the frequency $n = W/\lambda$ and prove that this agrees with the frequency of the condenser circuit which excites that oscillation. In the case of the helix above mentioned the measurement of this internodal distance for two consecutive nodes for the various harmonics was as follows: for the 1st harmonic 140 cms., for the 2nd harmonic 86 cms., for the 3rd harmonic 62 cms., for the 4th harmonic 48 cms., and for the 5th harmonic 39 cms. These distances are the half wave lengths. Hence, doubling them, we have 280, 172, 124, 96, and 78 for the harmonic series of observed wave lengths λ. Correspondingly it was necessary to adjust the condenser capacity C_1 and inductance L_1 so that the frequencies n calculated from the formula $n = \frac{1}{2\pi\sqrt{C_1L_1}}$ gave values respectively of

$0{\cdot}588 \times 10^6$ to produce the 1st harmonic,
$0{\cdot}977 \times 10^6$ to produce the 2nd harmonic,
$1{\cdot}379 \times 10^6$ to produce the 3rd harmonic,
$1{\cdot}70 \times 10^6$ to produce the 4th harmonic,
$1{\cdot}9 \times 10^6$ to produce the 5th harmonic.

Taking the observed values of the wave length λ and the calculated values of the frequency n, we can deduce the wave velocities $W = n\lambda$, and these are respectively 165×10^6, 168×10^6, 171×10^6, 163×10^6, and 148×10^6. The mean value is $163 \times 10^6 = W$. This compares fairly well with the calculated value 172×10^6 determined from the measured capacity and inductance of the helix per unit of length, having regard to the small value of these last quantities and consequent difficulty in measuring them exactly.

It is sufficient to show that all the harmonic oscillations travel with equal velocity, and that this velocity is equal to the value of $1/\sqrt{CL}$, where C and L are the capacity and inductance per unit of length of the helix.

The condition then for obtaining *stationary electric waves* on the helix is that the time taken for the wave to run *twice* to and fro on the helix must bear some integer ratio to the period of the applied electromotive force. If l is the length of the helix and W the wave velocity, then the time taken for the wave to run *twice* there and back along it is $4l/W$. But $W = 1/\sqrt{CL}$. Hence $t = 4l\sqrt{CL}$.

Suppose then that the time period of the applied electromotive force is $T = 4l\sqrt{CL}$, the wave will travel twice to and fro in this time, and we shall have the ratio $T/t=1$, or the oscillation excited will be the fundamental oscillation.

The wave length λ will then be such that $\lambda = WT = 4l$, or the fundamental wave length will be four times the length of the helix, or $4 \times 215 = 860$ cms.

If, however, the frequency of the applied electromotive force is three times greater, or $T_1 = \frac{4l}{3}\sqrt{CL}$, then the ratio $T_1/t = \frac{1}{3}$, and the wave length $\lambda_1 = WT_1 = \frac{4l}{3}$. If the frequency of the applied electromotive force is increased respectively to 5, 7, 9, 11, etc., times that required to create the fundamental oscillation, we shall have time periods $T_2 = \frac{4l}{5}\sqrt{CL}$, $T_3 = \frac{4l}{7}\sqrt{CL}$, $T_4 = \frac{4l}{9}\sqrt{CL}$, etc., and ratios $T_2/t = \frac{1}{5}$, $T_3/t = \frac{1}{7}$, etc., and therefore wave lengths $\lambda_2 = \frac{4l}{5}$, $\lambda_3 = \frac{4l}{7}$, $\lambda_4 = \frac{4l}{9}$, $\lambda_5 = \frac{4l}{11}$.

In the case of the helix described these harmonic wave lengths should therefore be 860/3, 860/5, 860/7, 860/9, 860/11 cms., or 286, 172, 123, 95, and 79 cms. respectively.

But the observed values as obtained from twice the internodal distances were 280, 172, 124, 96, and 78 cms. respectively, so the observed values of λ_2, λ_3, etc., agree very well with those which theory requires.

Hence any such helix of length l can have stationary waves produced upon it, fundamental or harmonic oscillations of wave length $\lambda_0 = 4l$, $\lambda_1 = \frac{4l}{3}$, $\lambda_2 = \frac{4l}{5}$, $\lambda_3 = \frac{4l}{7}$, $\lambda_4 = \frac{4l}{9}$, $\lambda_5 = \frac{4l}{11}$, etc., by applying to its end alternating electromotive forces of increasing frequency in the ratios 1, 3, 5, 7, 9, etc.

These facts have application in wireless telegraphy. An essential feature of the arrangements for producing the electric waves which are radiated through space to conduct wireless telegraphy is a long wire insulated at one end and connected to the earth or to a balancing capacity at the other end. The wire is called the aerial or antenna. At some point near the earthed end a high frequency electromotive force is applied in the wire,[1] and the frequency of this electromotive force is adjusted with reference to the length of the wire so as to produce stationary oscillations in the wire subject to the condition that the earthed or lower end must be a node of potential and the upper or insulated end of the wire a loop or antinode of potential. We can therefore set up oscillations which are the fundamental or higher harmonics, and which have frequencies in the ratio of 1, 3, 5, 7, 9, etc. These oscillations on the wire create electric waves in the space around. In the same manner we can set up on spiral wires stationary oscillations of various kinds. The possible types of oscillation on an aerial wire or antenna as used in radiotelegraphy are illustrated in Fig. 2, where the ordinates of the dotted line or its distance from the thick black line, representing the antenna, denotes the amplitude of the potential oscillation at that point in the wire.[2]

5. The Propagation of Currents along Leaky Lines.—Turning then to the fourth reduced case of the general equation, we have to discuss equation (4) for the case in which the frequency is very low, or the current even continuous, and the inductance and capacity small, but the resistance and leakance large. In this case, when the quantity pL can be

[1] For details see the Author's works on Wireless Telegraphy, "An Elementary Manual of Radiotelegraphy and Radiotelephony," or "The Principles of Electric Wave Telegraphy and Telephony" (Longmans, Green & Co., 39, Paternoster Row, London).

For further information on the production of stationary fundamental and harmonic oscillations in wireless telegraph antennæ the reader is referred to the Author's book "The Principles of Electric Wave Telegraphy and Telephony," Chapter IV., 2nd Edition.

neglected in comparison with R and also pC in comparison with S, the general equation reduces to

$$\frac{d^2v}{dx^2}=RSv.$$

Let us write a^2 for RS. Then the equation becomes

$$\frac{d^2v}{dx^2}=a^2v.$$

This is a well-known differential equation, which is satisfied by $v = A\epsilon^{ax}$ or $v = B\epsilon^{-ax}$, where A and B are constants. Hence the solution in the above case is

$$v=A\epsilon^{ax}+B\epsilon^{-ax}.$$

Instead of ϵ^{ax} and ϵ^{-ax} substitute in the above equation the equivalent expressions,

$$\epsilon^{ax}= \text{Cosh } ax+\text{Sinh } ax' \text{ and}$$
$$\epsilon^{-ax}= \text{Cosh } ax-\text{Sinh } ax.$$

We have then on collecting terms

$$v=(A+B) \text{ Cosh } ax+(A-B) \text{ Sinh } ax \quad . \quad . \quad (12)$$

If we take the origin at the sending end of the cable and assume that an electromotive force V_1 is applied at that end, then when $x = 0$ we have $v = V_1$, but when $x = 0$ Cosh $ax = 1$, Sinh $ax = 0$. Hence $V_1 = A + B$.

Again, the current i at any point in the line is equal to $-\frac{1}{R}\frac{dv}{dx}$, since the current is measured by the drop in potential down a length dx divided by the resistance of that length. If we differentiate (12) and multiply by $-\frac{1}{R}$ we have the expression for the current

$$i=-\frac{a}{R}(A+B) \text{ Sinh } ax-\frac{a}{R}(A-B) \text{ Cosh } ax \quad . \quad (13)$$

But when $x = 0$ $i = I_1$ = current at the sending end. Therefore we have

$$I_1=-\frac{a}{R}(A-B), \text{ or } A-B=-\frac{RI_1}{a}$$

and also $A + B = V_1$.

Substituting these values of $A + B$ and $A - B$ in (12), we have

$$v=V_1 \text{ Cosh } ax-\frac{RI_1}{a} \text{ Sinh } ax \quad . \quad . \quad . \quad (14)$$

and, since $i = -\frac{1}{R}\frac{dv}{dx}$, we find

$$i = I_1 \text{ Cosh } ax - \frac{V_1 a}{R} \text{ Sinh } ax \quad . \quad . \quad . \quad (15)$$

Let us denote the *insulation resistance* of the line per mile by r; then $r = 1/S$, and, since $a = \sqrt{RS}$, we have $a = \sqrt{\frac{R}{r}}$, and substituting this value of a in (14) and (15), we arrive finally at the expressions

$$v = V_1 \text{ Cosh } ax - I_1 \sqrt{Rr} \text{ Sinh } ax \quad . \quad . \quad (16)$$

$$i = I_1 \text{ Cosh } ax - \frac{V_1}{\sqrt{Rr}} \text{ Sinh } ax \quad . \quad . \quad . \quad (17)$$

which give us the potential v and current i at any distance x from the sending end of a line of conductor resistance R and insulation resistance r per unit of length.

We will then consider various cases in which the line is (i.) insulated, (ii.) earthed at the far end, and (iii.) earthed through a receiving instrument of known resistance.

(i.) *Line insulated at the far end.*—In this case we have zero current at the extremity. Hence in equation (17) put $i = 0$ and $x = l$, where l is the length of the line; then

$$I_1 \text{ Cosh } al = \frac{V_1}{\sqrt{Rr}} \text{ Sinh } al \quad . \quad . \quad . \quad (18)$$

or

$$I_1 \sqrt{Rr} = V_1 \text{ Tanh } al \quad . \quad . \quad . \quad . \quad (19)$$

Substituting from equation (19) in (16), we have

$$v = V_1\{\text{Cosh } ax - \text{Sinh } ax \text{ Tanh } al\} \quad . \quad . \quad (20)$$

This gives us the potential v at any point in a leaky line.

If we take $x = l$, then (20) becomes

$$v = V_1 \text{ Sech } al \quad . \quad . \quad . \quad . \quad (21)$$

and as l increases v continually diminishes.

If the line had no leakage, that is if $r = \infty$, then we should have had $v = V_1$ at the far end when that end is insulated.

Also from (19) and (17) we find

$$i = I_1\{\text{Cosh } ax - \text{Sinh } ax \text{ Coth } al\} \quad . \quad . \quad . \quad (22)$$

which gives us the current at any point in the leaky line.

We can put the formulæ (20) and (22) for the voltage and current in a simpler form if we measure the distances from the free end. Let x' be the distance of a point from the free end, and let $x' = l - x$.

Then formula (20) is equivalent to

$$v=\frac{V_1}{\text{Cosh } al}\text{ Cosh } ax' \quad . \quad . \quad . \quad . \quad (23)$$

and (22) can be written

$$i=\frac{I_1}{\text{Sinh } al}\text{ Sinh } ax' \quad . \quad . \quad . \quad . \quad (24)$$

Hence the potential at any point in the leaky line is proportional to the hyperbolic cosine of ax' and the current to the hyperbolic sine of ax'. Hence when $x' = 0$ we have

$$v=V_1/\text{Cosh } al=V_1 \text{ Sech } al,$$

as before. Let us consider next,

(ii.) *The line earthed at the far end.*—Then for $x = l$ we have $v = 0$, and therefore substituting these values in (16), we have

$$I_1\sqrt{Rr}\text{ Sinh } al=V_1 \text{ Cosh } al \quad . \quad . \quad . \quad (25)$$

and substituting this last, (25), in both (16) and (17), we arrive at the equations

$$v=V_1\{\text{Cosh } ax-\text{Sinh } ax \text{ Coth } al\} \quad . \quad , \quad (26)$$

$$i=I_1\{\text{Cosh } ax-\text{Sinh } ax \text{ Tanh } al\} \quad . \quad . \quad (27)$$

If we reckon distances from the earthed end and let x' be such distance, so that $x' = l - x$, then, substituting in the above formulæ, we have

$$v=\frac{V_1}{\text{Sinh } al}\text{ Sinh } ax' \quad . \quad . \quad . \quad . \quad (28)$$

$$i=\frac{I_1}{\text{Cosh } al}\text{ Cosh } ax' \quad . \quad . \quad . \quad . \quad (29)$$

Hence at the earthed or receiving end the current is given by

$$i=I_1 \text{ Sech } al . \quad . \quad . \quad . \quad . \quad (30)$$

and when l is very large this received current is zero.

We have then to consider the case

(iii.) *When the line is earthed through a receiving instrument of known resistance.*—We shall consider that the receiving instrument has a resistance ρ and a negligible inductance. Then the current through the receiving instrument is $I_2 = V_2/\rho$.

Referring to the general equations (16) and (17),

$$v = V_1 \operatorname{Cosh} ax - I_1 \sqrt{Rr} \operatorname{Sinh} ax,$$

$$i = I_1 \operatorname{Cosh} ax - \frac{V_1}{\sqrt{Rr}} \operatorname{Sinh} ax,$$

we put $x = l$, and we have

$$V_2 = I_2 \rho = V_1 \operatorname{Cosh} al - I_1 \sqrt{Rr} \operatorname{Sinh} al \quad . \quad . \quad (31)$$

$$I_2 = I_1 \operatorname{Cosh} al - \frac{V_1}{\sqrt{Rr}} \operatorname{Sinh} al \quad . \quad . \quad . \quad (32)$$

Eliminating I_1 from these two last equations we obtain

$$I_2 = \frac{V_1}{\rho \operatorname{Cosh} al + \sqrt{Rr} \operatorname{Sinh} al} \quad . \quad . \quad (33)$$

Also eliminating I_2, we have

$$I_1 = \frac{V_1}{\sqrt{Rr}} \cdot \frac{\sqrt{Rr} \operatorname{Cosh} al + \rho \operatorname{Sinh} al}{\rho \operatorname{Cosh} al + \sqrt{Rr} \operatorname{Sinh} al} . \quad . \quad . \quad (34)$$

Consider a hyperbolic angle γ such that $\operatorname{Tanh} \gamma = \rho / \sqrt{Rr}$, and therefore $\operatorname{Sinh} \gamma = \frac{\rho}{\sqrt{Rr - \rho^2}}$, and $\operatorname{Cosh} \gamma = \frac{\sqrt{Rr}}{\sqrt{Rr - \rho^2}}$.

Then we can write the expressions (33) and (34) in the form

$$I_2 = \frac{V_1}{\sqrt{Rr - \rho^2}} \operatorname{Cosech} (al + \gamma) \quad . \quad . \quad (35)$$

$$I_1 = \frac{V_1}{\sqrt{Rr}} \operatorname{Coth} (al + \gamma) . \quad . \quad . \quad . \quad (36)$$

On comparing the above expressions with those given in Chapter III. for the propagation of telephone currents in a line with constants R, L, C, and S, it will be seen that the expressions are similar, but that the quantity $\sqrt{Rr}$ here takes the place of the initial sending end impedance and ρ that of the impedance of the receiving instrument.

The ratio of the received to the sending end current is

$$\frac{I_2}{I_1} = \frac{\sqrt{Rr}}{\sqrt{Rr - \rho^2}} \operatorname{Sech} (al + \gamma) \quad . \quad . \quad . \quad (37)$$

which reduces to (30) when $\rho = 0$. All these expressions are applicable to continuous currents flowing in leaky lines. For a given line of given leak per mile the effect of placing a receiving instrument at the receiving end is equivalent to increasing the length of the line by an amount l' such that

$$al' = \gamma \text{ or } l' = \sqrt{\frac{r}{R}} \operatorname{Tan}^{-1} \frac{\rho}{\sqrt{Rr}}.$$

CHAPTER VII

ELECTRICAL MEASUREMENTS AND DETERMINATION OF THE CONSTANTS OF CABLES

1. Necessity for the Accumulation of Data by Practical Measurements.—As a long submarine cable or telephone line is a costly article, the predetermination of its performance is a matter of the utmost importance. It is therefore necessary to bring to bear upon its construction and testing a large knowledge of the results of previous constructions of the same or similar cables. This requires electrical testing. In fact, we may say that out of the attempts to lay the first very long submarine cables the whole of our practical and absolute system of electrical measurements has arisen. We have to determine for every cable and line the primary constants, viz., conductor resistance, inductance, capacity, and the insulation resistance, all per statute or nautical mile or kilometre, and especially measurements of the attenuation constants, to provide a store of knowledge on which we can draw in designing other cables. Experimental means are therefore required for accurately measuring these quantities as well as others, such as line and instrumental impedances, and the currents and phase angles to enable forecasts to be made of the operation of proposed lines or cables when constructed in a predetermined manner. For much of the information on the methods of electrical measurements generally the reader must be referred to existing text-books, but it will be convenient to epitomise some of the most necessary information in this chapter.[1]

[1] The reader may be referred to a treatise by the Author entitled "A Handbook for the Electrical Laboratory and Testing Room," 2 vols., *The Electrician* Printing and Publishing Company, Ld., 1, Salisbury Court, Fleet Street, and also to the well-known work by Mr. H. R. Kempe on "Electrical Testing."

2. The Predetermination of Capacity.—Since a telegraph or telephone wire is only a long cylinder of metal or else a similar structure composed of stranded wires of which the section is approximately circular, we have first to consider the capacity of such a long cylinder in various positions with regard to the earth or other conductors.

Definition.—The electrical capacity of a body is measured by the quantity of electricity or charge which must be imparted to it to raise its potential by one unit when all other neighbouring conductors are maintained at zero potential.

Definition.—The potential at any point due to any charge on an extremely small conductor at any other point is measured by the quotient of the small charge or quantity of electricity by the distance between the conductor and the point in question. Hence if we have any small charge dq on a conductor the potential at a distance r from that charge is dq/r. The potential due to a finite charge is the sum of all the potentials due to the elements of the charge respectively. Thus if a body has a charge Q, and we divide it into elements of charge dQ, then the potential at any point is the sum of all the quantities dQ/r, where r is the distance from the point in question to each element of the total charge.

Two other facts connected with electric potential and charge are (i.) that electric charge resides only on the surface of conductors, and (ii.) that the potential of all parts of a conductor is the same. These principles enable us to calculate the capacity of conductors of a certain symmetry of form in simple cases. For example, we may find the capacity of a conducting sphere as follows: Let a charge Q be supposed to be uniformly distributed over it, and let it be assumed to be divided into elements of charge dQ. Let the radius of the sphere be R. Then the potential at the centre of the sphere due to each element of charge is dQ/R, and, since all elements are situated similarly with regard to the centre of the sphere, the potential at the centre of the whole charge is Q/R. But this must therefore be the potential V of any point in the sphere. Hence $Q/R = V$ or $Q/V = R$. Now the ratio of charge to potential is defined to be the capacity C of the conductor. Hence for such a sphere $C = R$, or the capacity in electrostatic units is numerically equal to the radius of the sphere.

Since 9×10^5 electrostatic units capacity are equal to 1 microfarad, we find that the capacity of the sphere of radius R is equal to $R/(9 \times 10^5)$ microfarads, where R is measured in centimetres.

This, however, is on the assumption that the sphere has a uniformly distributed charge, and that all other conductors are at a very great distance. The actual capacity of a conducting sphere of radius R cms. hung up in a room, for instance, would be found to be somewhat more than $R/(9 \times 10^5)$ microfarads.

For instance, let a conducting sphere be surrounded by a concentric spherical shell, and let the radius of the outer surface of the inner sphere be R_1 and that of the inner-surface of the outer shell be R_2. Then if a positive charge Q is placed on the inner sphere it will induce an equal negative charge on the inner surface of the outer shell, and if this outer shell is earthed the potential at any point in the inner sphere will be $\frac{Q}{R_1} - \frac{Q}{R_2} = V$, and hence $\frac{Q}{V} = C = \frac{R_1 R_2}{R_2 - R_1}$ electrostatic units, or the capacity of the inner sphere in microfarads will be $\frac{R_1 R_2}{R_2 - R_1} \frac{1}{9 \times 10^5}$ mfds., which becomes equal to $R_1/(9 \times 10^5)$ when R_2 is infinite. The capacity of the sphere is therefore increased by the proximity of another conductor even though the latter is connected to earth.

In the same manner we can obtain an expression for the capacity of a long cylindrical wire of circular section. Take a point O on the central axis for origin, and consider any element of the surface cut off by two transverse planes. Let the radius of the circular section be r, and the axial length of the element be δx, and the axial distance of the elements from the origin be x. Then the surface of that element is $2\pi r \delta x$, and if ρ is the surface density of a charge uniformly distributed over the wire, the charge on that element of surface is $2\pi r \rho \delta x$. The distance of all parts of this element of charge from the origin is $\sqrt{r^2 + x^2}$, and hence the potential of the element at the origin is

$$dV = \frac{2\pi r\rho\delta x}{\sqrt{r^2+x^2}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence the potential V of the whole charge spread uniformly over a wire of length l is obtained from the integral

$$V = 2\int_0^{\frac{l}{2}} \frac{2\pi\rho r\delta x}{\sqrt{r^2+x^2}} \quad . \quad . \quad . \quad . \quad (2)$$

The integral $\displaystyle\int \frac{dx}{\sqrt{r^2+x^2}} = \log_\epsilon \left\{x + \sqrt{r^2+x^2}\right\}$.

Hence $\displaystyle V = 4\pi r\rho \left\{ \log_\epsilon \left(\frac{l}{2} + \sqrt{r^2 + \frac{l^2}{4}}\right) - \log_\epsilon r \right\}$ . . (3)

But, since $Q = 2\pi r\rho l$ is the whole charge on the wire, the capacity $C = Q/V$. Therefore we have for the capacity of the circular-sectioned wire of length l and diameter $d = 2r$ the expression

$$C = \frac{l}{2\left\{ \log_\epsilon \left(\frac{l}{2} + \sqrt{r^2 + \frac{l^2}{4}}\right) - \log_\epsilon r \right\}} \quad . \quad . \quad . \quad (4)$$

and if r is small compared with $\frac{l}{2}$ this becomes

$$C = \frac{l}{2\log_\epsilon \frac{2l}{d}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The above formula gives the capacity in electrostatic units. If we use ordinary logarithms and reckon in microfarads it becomes

$$C \text{ (in mfds.)} = \frac{l}{4{\cdot}6052 \times 9 \times 10^5 \times \log_{10} \frac{2l}{d}} \quad . \quad . \quad (6)$$

The length l must be expressed in centimetres.

This formula is useful in calculating the capacity of a single vertical wire used as an antenna in radiotelegraphy, but in practice it will generally give a value about 10 per cent. or so, too small on account of the proximity of the antenna wire to the earth. The formula (4) is in fact the capacity of a wire at an infinite distance from all other conductors.

Another useful expression for the potential of a long, straight, thin-charged wire at a point outside the wire may be obtained as follows: Let P be the point and PO a perpendicular let fall on the wire. Take O as origin and measure off any distance x (see Fig. 1) along the wire. Let δx be an element of length at this distance, and let the charge on the wire be q electrostatic

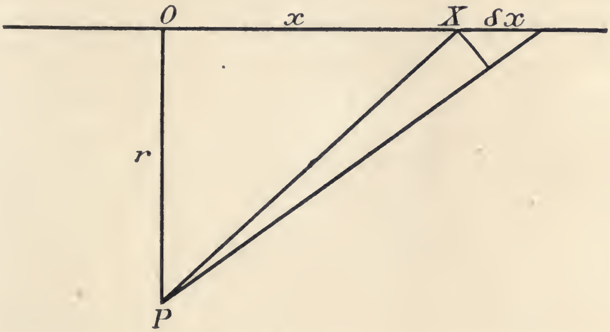

FIG. 1.

units per unit of length of the wire. Then the electric force due to the charge $q\delta x$ on δx at P in the direction PO is

$$dF = \frac{rq\delta x}{\left(r^2+x^2\right)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad (7)$$

where r is the length PO.

Hence the electric force at P due to the whole charge on the infinitely long wire resolved in the direction PO is

$$F = 2\int_0 \frac{rq\delta x}{\left(r^2+x^2\right)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad (8)$$

But

$$\int \frac{r\delta x}{\left(r^2+x^2\right)^{\frac{3}{2}}} = \frac{x}{r\sqrt{r^2+x^2}} \quad . \quad . \quad . \quad . \quad (9)$$

Hence

$$F = \frac{2q}{r} = -\frac{dV}{dr}$$

since the force F is the rate of decrease of the potential V at P in the direction of F.

Accordingly we have $$-\frac{dV}{dr}=\frac{2q}{r},$$

or $$dV=-2q\frac{dr}{r}.$$

Hence, integrating this last equation, we have
$$V=-2q\log_\epsilon r+C \quad . \quad . \quad . \quad . \quad (10)$$
where C is some constant of integration. Availing ourselves of this expression, we can obtain approximate expressions for the capacity of aerial telegraph and telephone wires.

3. The Capacity of Overhead Telegraph Wires.—Consider the case of two long circular sectioned wires stretched parallel to each other with their centres at a distance D which is large compared with the diameter of the wires. If then this distance is sufficiently large to prevent the charge on each wire disturbing the uniformity of distribution of the charge on the other wire we may consider that the charge on each wire is uniformly distributed round the surface and equivalent to a number of uniformly electrified filaments arranged on the surface of a cylinder parallel to its axis.

Let one wire be denoted by A and be supposed to be charged positively and the other wire be B and be charged negatively. Then the potential at the centre of A may be denoted by V_A, and bearing in mind the expression for the potential of a filament at any point outside it, it will be clear that this potential V_A is given by
$$V_A=(-2q\log r+C)-(-2q\log D+C) \quad . \quad . \quad (11)$$
because the distance of all the charge on A from the centre of A is r and the distance of all the charge on B from the centre of A is nearly D.

Similarly the potential V_B at the centre of B is
$$V_B=-(-2q\log r+C)+(-2q\log D+C) \quad . \quad . \quad (12)$$
and hence
$$V_A-V_B=4q(\log_\epsilon D-\log_\epsilon r)=4q\log_\epsilon\frac{D}{r} \quad . \quad . \quad (13)$$

But the charge per unit of length of the wires is q, and their difference of potential is $V_A - V_B$, therefore the capacity per unit of length C is $q/(V_A-V_B) = \dfrac{1}{4\log_\epsilon\frac{D}{r}}$, electrostatic units.

Accordingly the mutual capacity for a length l cms. of the two wires, each of diameter d cms. and distance D cms., where D is large compared with d, is given in microfarads by the expression

$$C \text{ (in mfds.)} = \frac{l}{4 \times 2{\cdot}3026 \times 9 \times 10^5 \times \log_{10} \frac{2D}{d}} \quad . \quad . \quad (14)$$

The factor 2·3026 is the multiplier for converting logarithms to the base 10 to Napierian logarithms. The above reduces to

$$C \text{ (in mfds.)} = \frac{0{\cdot}0000001208 l}{\log_{10} \frac{2D}{d}} \quad . \quad . \quad . \quad (15)$$

Since 1 mile = 160934·4 cms., the capacity per mile of two such parallel wires at a distance D is

$$C \text{ (in mfds.)} = \frac{0{\cdot}0194}{\log_{10} \frac{2D}{d}} \quad . \quad . \quad . \quad . \quad (16)$$

provided D is large compared with d and the wires are both very high above the earth.

If the wires are at all close together the capacity per unit of length is greater than that given by the above formulæ. The mutual attractions disturb the uniform perimetral distribution of the charges, and the calculation of the capacity becomes much more difficult.

In ordinary overhead telephone wires the lead and return will generally be sufficiently far apart to make the formulæ approximately correct, but for twin wires enclosed in the same insulating sheath where the wires are not more than two or three diameters apart the above formulæ are not sufficiently correct to do more than give an approximation. Moreover, in the latter case the expressions for the capacity have to be multiplied by a factor called the dielectric constant, or specific inductive capacity of the dielectric.

A derivative case of the above is that of a single wire placed parallel to, and at a height h above, the surface of the earth.

If we suppose the earth's surface to be a good conductor and at zero potential, then the difference of potential between the charged wire at a height h above the earth and the earth would be half of that between the charged wire and a similar oppositely charged wire at a depth h below the surface of the earth, supposing all the earth then removed. Hence the capacity of the single wire at a height h above the earth must be double that of two parallel wires at distance $2h$ apart. Accordingly the capacity of a length l of telegraph wire parallel to the earth and at a height h above it is $C=\dfrac{2l}{4\log_\epsilon\dfrac{4h}{d}}$ electrostatic units, where d is the diameter of the wire.

In microfarads we have

$$C\text{ (in mfds.)}=\frac{l}{2\times 2{\cdot}3026\times 9\times 10^5\times\log_{10}\dfrac{4h}{d}} \quad . \quad . \quad (17)$$

and the capacity per mile in microfarads is given by

$$C\text{ (in mfds.)}=\frac{0{\cdot}0388}{\log_{10}\dfrac{4h}{d}} \quad . \quad . \quad . \quad . \quad (18)$$

A rather more accurate formula is given in *The Electrician* for January 28th, 1910, p. 645. It is

$$C\text{ (in electrostatic units)}=\frac{l}{2\log_\epsilon\dfrac{h+\sqrt{h^2+r^2}}{r}} \quad . \quad . \quad (19)$$

where r is the radius of the section of the wire.

4. The Capacity of Concentric Cylinders and of Submarine Cables.—The next important case is that of the capacity of a pair of concentric cylinders.

Let us suppose a conducting cylinder having a circular cross section of radius R_1 to be placed concentrically in the interior of a conducting cylinder of inner radius R_2. Let the inner cylinder be charged with positive electricity. Then this will induce an equal negative charge on the inner surface of the outer cylinder, and we shall assume that this outer cylinder is connected to earth. These charges may be considered to be made up of filamentary charges laid along the surfaces.

Let the cylinders be so long that the effect of the end distributions may be neglected, and let the charge per unit of length on the inner or outer cylinder be q electrostatic units. Then, since all the filamentary charges are at the same distance from the centre, the potential at the centre of the inner cylinder, which we shall call V, is given by

$$V=(-2q \log_\epsilon R_1+C)-(-2q \log_\epsilon R_2+C),$$

or

$$V=2q \log \frac{R_2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

But the whole charge on the cylinders, assuming them to have a length l and supposing the irregularity in distribution at the ends to be neglected, is $ql = Q$.

The capacity per unit of length of the cylinders is then $q/V = C$, and

$$C=\frac{1}{2 \log_\epsilon \frac{R_2}{R_1}} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If the capacity is reckoned in microfarads and ordinary logarithms used we have

$$C \text{ (in mfds.)}=\frac{1}{2 \times 2{\cdot}3026 \times \log_{10} \frac{R_2}{R_1} \times 9 \times 10^5} \quad . \quad . \quad (22)$$

If the dielectric used between the cylinders has a dielectric constant K, then the capacity for a length l is

$$C \text{ (in mfds.)}=\frac{Kl}{4{\cdot}6052 \times 9 \times 10^5 \times \log_{10} \frac{R_2}{R_1}} \quad . \quad . \quad (23)$$

Since 1 mile = 160934·4 cms., and since the constant

$$\frac{160934{\cdot}4}{4{\cdot}6052 \times 9 \times 10^5}=0{\cdot}0388,$$

we have for the capacity per mile the expression

$$C \text{ (in mfds.)}=\frac{0{\cdot}0388K}{\log_{10} \frac{R_2}{R_1}} \quad . \quad . \quad . \quad (24)$$

where K is the dielectric constant. For gutta-percha $K = 2{\cdot}46$, for india-rubber (pure) $K = 2{\cdot}12$, for india-rubber (vulcanised) $K = 2{\cdot}69$, and for paper insulation $K =$ about 1·25 or less.

5. Formulæ for the Inductance of Cables.—The inductance of a circuit is that quality of it in virtue of which energy is associated with the circuit when a current exists in it. It is defined numerically by the total magnetic flux or total number of lines of magnetic flux which are linked with the circuit when unit current flows in it and when no other currents or magnetic fields are in its neighbourhood. The creation of the magnetic field embracing a circuit when an electric current is started in it, requires the expenditure of energy, and as long as it exists it represents a store of energy.

This energy is measured by $\frac{1}{2}Li^2$, where i is the current and L is the inductance of the circuit. This is proved in the following manner:

If an electromotive force v is applied to a circuit and creates in it a current i, and if this state of affairs endures for a small time dt, then the work done on the circuit is $vi\,dt$. If the circuit has a resistance R the energy dissipated in it by resistance is Ri^2dt, and hence the difference $(vi - Ri^2)dt$ must represent the energy stored up in connection with the circuit in the time dt. The expression may be written $(v - Ri)idt$, and therefore $v - Ri$ must be a counter-electromotive force created in the circuit as the current increases in it. By Faraday's law of induction the electromotive force must be measured by the time rate of increase of the total self-linked magnetic flux. Let L be the inductance of the circuit; then Li is the self-linked magnetic flux when a current i exists in the circuit, and therefore $L\frac{di}{dt}$ must be the counter-electromotive force due to the variation of this self-linked flux. Accordingly we have the equation

$$L\frac{di}{dt}=v-Ri,$$

or

$$L\frac{di}{dt}+Ri=v \qquad . \quad . \quad . \quad . \quad (25)$$

as the differential equation connecting the current in the circuit i with the impressed electromotive force v at any instant.

Also the energy stored up in connection with the circuit in a time dt must be $L\frac{di}{dt}\,i\,dt = Li\,di$, and in establishing a current which starts from zero and reaches a final value I the total energy stored up must be equal to

$$\int_0^I Li\,di=\frac{1}{2}LI^2.$$

If L is a certain coefficient or number called the inductance of the circuit, then when a current i flows in the circuit the total magnetic flux produced which is self-linked with the circuit is measured by Li. The total energy associated with the circuit is measured by $\frac{1}{2}Li^2$, and the counter-electromotive force due to the variation of this self-linked flux is measured by $L\frac{di}{dt}$.

The quantity L, or the inductance, is measured in terms of a unit called *one henry*, and since the dimensions of this quantity in electromagnetic measure are those of a length, the absolute electromagnetic measurement of inductance is expressed in centimetres. The calculation of the inductance of a circuit is effected by ascertaining the potential energy associated with two

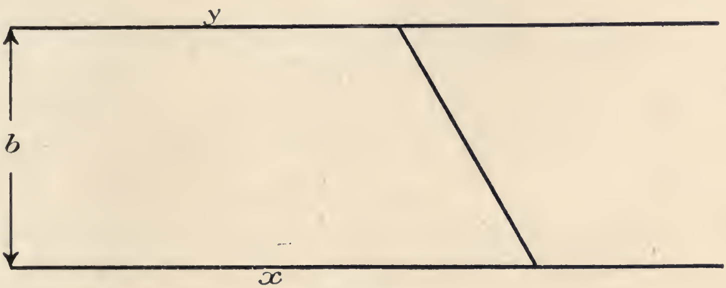

Fig. 2.

similar circuits when unit current flows in each, and the circuits are placed parallel and at a certain distance apart. This may be accomplished by means of a formula due to Neumann, the proof of which is to be found in many advanced text-books on electrical theory. It is as follows: Let ds and ds' be elements of length, one in each of the two circuits, and let θ be the angle between their direction, and r the distance between them. Then the mutual inductance M of the two circuits can be found by taking the integral

$$M=\iint\frac{\operatorname{Cos}\theta}{r}\,ds\,ds' \quad . \quad . \quad . \quad . \quad (26)$$

where the integration is extended to every possible pair of elements.

Suppose, for instance, we consider two very thin, straight parallel wires of length l placed at a distance b apart. Then, taking the origin at the end of each wire, we define one element, dx, in one wire by its distance x from the origin, and the other element, dy, by its distance y from the other origin. The distance apart of these elements is $\sqrt{(x-y)^2+b^2}$, and their inclination is zero. Hence Cos $\theta = 1$ (see Fig. 2).

The mutual induction is then given by

$$M=\int_o^l\int_o^l \frac{dx\ .\ dy}{\sqrt{(x-y)^2+b^2}} \quad . \quad . \quad . \quad . \quad (27)$$

The integral $\displaystyle\int\frac{dx}{\sqrt{(x-y)^2+b^2}}=\log\{x-y+\sqrt{(x-y)^2+b^2}\}$,

and hence $\displaystyle\int_o^l\frac{dx}{\sqrt{(x-y)^2+b^2}}=\log\left\{\frac{l-y+\sqrt{(l-y)^2+b^2}}{-y+\sqrt{y^2+b^2}}\right\}$ . . (28)

Again,
$$\int\log\{(l-y)+\sqrt{(l-y)^2+b^2}\}\,dy$$
$$=-(l-y)\log\{(l-y)+\sqrt{(l-y)^2+b^2}\}+\sqrt{(l-y)^2+b^2} \quad . \quad (29)$$

and
$$\int\log\{-y+\sqrt{y^2+b^2}\}\,dy$$
$$=y\log\{-y+\sqrt{y^2+b^2}\}+\sqrt{y^2+b^2} \quad . \quad . \quad (30)$$

Hence

$$M=\int_o^l\int_o^l\frac{dx\,dy}{\sqrt{(x-y)^2+b^2}}=\int_o^l\log\left\{\frac{l-y+\sqrt{(L-y)^2+b^2}}{-y+\sqrt{y^2+b^2}}\right\}dy \quad . \quad (31)$$

and
$$M=l\log\left\{\frac{l+\sqrt{l^2+b^2}}{-l+\sqrt{l^2+b^2}}\right\}-2\sqrt{l^2+b^2}+2b \quad . \quad . \quad (32)$$

Since
$$\frac{l+\sqrt{l^2+b^2}}{-l+\sqrt{l^2+b^2}}=\left\{\frac{l+\sqrt{l^2+b^2}}{b}\right\}^2,$$

we can write $\displaystyle M=2\left\{l\log\left(\frac{l+\sqrt{l^2+b^2}}{b}\right)-\sqrt{l^2+b^2}+b\right\}$. (33)

and if b is small compared with l this reduces to

$$M=2l\left\{\log\frac{2l}{b}-1\right\} \quad . \quad . \quad . \quad . \quad (34)$$

or
$$M=2l\log 2l-2l-2l\log b.$$

Therefore the expression for M if l is constant and b varies is of the form

$$M=A-B\log b \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where A and B are constants, and the logarithms are Napierian.

The above formulæ apply to the case of a pair of infinitely thin or filamentary currents. In the case of actual conductors we have the current distributed over a finite area or circumference. We may either have the current uniformly distributed over the cross section of the conductor, as in the case of steady or of low frequency currents, or we may have it distributed over the surface of the conductor or round the periphery, as in the case of high frequency currents. If then we deal with a pair of parallel wires of finite section we must consider the actual current as made up of filamentary currents either laid round the circumference of the wire or closely packed together uniformly over the cross section. In any case we shall have to obtain the actual mutual inductance by taking the mean value of a number of expressions such as $M = A + B \log b$, where the b applies to the perpendicular distance of a pair of selected filaments, one in one wire and the other in the other wire. The final result will be that in place of b we shall have a certain distance R such that $\log R$ is the mean value of all the values of $\log b$ for all possible pairs of filaments. If

$$\log R = \frac{1}{n}(\log b_1 + \log b_2 + \log b_3 + \text{etc.}),$$

then

$$R = (b_1 . b_2 . b_3 .)^{\frac{1}{n}} \quad . \quad . \quad . \quad . \quad (36)$$

and R is called the geometric mean of b_1, b_2, b_3, etc.

Hence the mutual inductance of two wires of finite section and length l is given by the expression

$$M = 2l\left\{\log \frac{2l}{R} - 1\right\} \quad . \quad . \quad . \quad . \quad (37)$$

where R is the geometric mean distance (*G.M.D.*) of all possible filamentary elements into which we can divide the currents, one being taken in one wire and one in the other.

The determination of this *G.M.D.* is a purely mathematical operation, and it can be shown that if the current is distributed over the surface of a circular-sectioned wire, as it is in the case of very high frequency currents, we have to find the *G.M.D.* of all possible pairs of elements, in the circumference of two circles, whilst if the current is a low frequency or continuous current we have to find the *G.M.D.* of all elements of area in the cross section of the two wires, one element being taken in or on each wire.

By the self-induction or inductance of a circuit we mean the inductance of the circuit on itself or the total flux per unit of current which is self-linked with the circuit. Hence to calculate the inductance of a straight wire we apply the above formula, but the quantity R becomes the *G.M.D.* of all the elements of current in that conductor itself.

If the current is a high frequency current or confined to the surface, say, of a circular-sectioned wire, we have then to find the *G.M.D.* of all possible pairs of points on the circumference of a circle, and Maxwell has shown that if d is the diameter of this circle, then the *G.M.D.* of all pairs of elements of the circumference is $\frac{d}{2}$.[1]

If, however, the current is a direct or low frequency current, then we have to find the *G.M.D.* of all possible elements of the cross-sectional area; and if the cross section is a circle, Maxwell has shown that this *G.M.D.* is equal to $\frac{d}{2}\epsilon^{-\frac{1}{4}} = \frac{d}{2} \times 0{\cdot}7788$, where ϵ is the base of the Napierian logarithms. Hence if we have a single straight wire of circular section, diameter d and length l, its inductance L is found by substituting in the formula

$$2l\left\{\log\frac{2l}{b} - 1\right\},$$

for the value of b either $b = \frac{d}{2}$ or $b = \frac{d}{2}\epsilon^{-\frac{1}{4}}$ according as the current is assumed to be distributed over the surface only or over the whole cross section.

For the kind of wires and for the frequencies with which we are concerned in telegraphy we may generally assume that the current is distributed uniformly over the cross section of a circular wire, and hence, putting $b = \frac{d}{2}\epsilon^{-\frac{1}{4}}$, we have

$$L = 2l\left\{\log\frac{4l}{d} - \frac{3}{4}\right\} \quad . \quad . \quad . \quad . \quad (38)$$

as the expression for the inductance of a wire of diameter d and length l. For high frequency currents the constant $\frac{3}{4}$ is replaced by 1.

[1] See Maxwell, "Treatise on Electricity and Magnetism," 2nd Ed., Vol. II., p. 298, § 691.

The above formula (38) enables us to calculate the inductance per unit of length of an overhead telephone wire provided it is made of non-magnetic material and is sufficiently far removed from all other wires.

It cannot, however, be applied to a wire made of iron or to a submarine telegraph cable in which a single stranded insulated copper wire is enclosed in steel armour, since in these cases the magnetic permeability of the iron increases the inductance by a certain unknown amount very difficult to predict.

In the case of a pair of parallel wires, if the wires are not so near that the distribution of current over the cross section of the wires is disturbed or if the wires are very thin we can calculate the inductance as follows: If one of these wires is a lead and the other a return, then their inductance is defined to be the magnetic flux per unit of current which is self-linked with this circuit. It is therefore equal to twice the difference between the mutual induction of the two wires when close together and when separated by a distance D.

If we consider a circular-sectioned wire of diameter d to have a filamentary conductor placed close to it and therefore at a mean distance $\frac{d}{2}$ the mutual inductance is equal to $A - 2l \log \frac{d}{2}$. If then the filament is removed to a distance D the mutual inductance is equal to $A - 2l \log D$.

Accordingly the self-induction or inductance is equal to twice the difference, or to $4l \log \frac{2D}{d}$.

The formula holds good approximately for a pair of wires of small diameter parallel to each other. Hence

$$L = 4l \log_\epsilon \frac{2D}{d},$$

or

$$L = 9{\cdot}2104 l \log_{10} \frac{2D}{d} \quad . \quad . \quad . \quad . \quad (39)$$

gives us a rough expression for the inductance of a length l of a pair of parallel wires each of diameter d with their axes separated by a distance D. All lengths must be measured in centimetres, and the inductance is then in centimetres, and must be divided by 10^9 to reduce it to henrys. An expression for the inductance of a concentric cable is sometimes required. Let us suppose that two conducting tubes are placed concentrically, and that the space between the two is filled with some dielectric. If the tubes are made of non-magnetic material, and if R_1 and R_2 are the radii of the inside and outside of the inner tube and R_3 and R_4 are the inner and outer radii of the outer tube, then Lord Rayleigh has shown that the inductance per unit of length of such a conductor is given by the expression

$$L=2\log\frac{R_3}{R_2}+\frac{2}{R_2^2-R_1^2}\left\{\frac{R_2^2-3R_1^2}{4}+\frac{R_1^4}{R_2^2-R_1^2}\log\frac{R_2}{R_1}\right\}$$
$$+\frac{2}{R_4^2-R_3^2}\left\{\frac{R_3^2-3R_4^2}{4}+\frac{R_4^2}{R_4^2-R_3^2}\log\frac{R_4}{R_3}\right\} \quad . \quad . \quad (40)$$

The logarithms are Napierian.

If the inner conductor is a solid rod of radius R_2, then R_1 is zero, and the expression becomes somewhat simplified, since then the first two terms become $2\log\frac{R_3}{R_2}+\frac{1}{2}$, and the third term comes in as a correcting factor.

6. The Practical Measurement of the Capacity of Telegraph and Telephone Cables.—We shall not attempt to discuss all the various methods which have been proposed or used for measuring the capacity of cables. The difficulties with which this measurement is attended depend chiefly upon the fact that when an electric force is applied to a dielectric the displacement which takes place is not merely a function of the force and nature of the dielectric, but also of the time of application of the force and its mode of variation. Thus if the electric force is applied and kept steadily applied the displacement increases very rapidly at first and afterwards moves slowly, and even after a long time there is a slow increase in the displacement, which may be only a true dielectric current or may be a conduction current superimposed on the dielectric current.

The conduction current is, however, distinguished from the dielectric current by the fact that the energy absorbed in creating it is dissipated as heat in the dielectric and is not recoverable, whilst the energy taken up in producing the true dielectric current is recovered in the discharge current when the condenser is short-circuited.

Nevertheless there is a considerable difference between the instantaneous or the high frequency capacity of a condenser and its capacity with steady unidirectional electric force applied continuously. The latter is considerably larger than the former for some dielectrics.

In the case of telephone cables the capacity with which we are concerned is that which corresponds to a frequency n of the electric force of about 800 or 750, or say for which $2\pi n = 5{,}000$.

In the case of submarine cables or low frequency alternating current power supply we may consider that the steady capacity is the more important.

Full discussion will be found in good text-books on electrical measurements concerning the various methods of measuring the capacity of cables with steady or low frequency alternating electric force. We shall here only refer to one method which enables us to measure the capacity of a cable for telephonic frequencies if necessary.

This method is that known as the commutator method. The length of cable to be tested is charged with a battery of a certain electromotive force and then discharged through a galvanometer. This process is repeated one hundred or several hundred times per second by means of a revolving commutator, and the successive discharges are sent through a galvanometer. This practically constitutes a continuous current the value of which in fractions of an ampere can be ascertained by employing the same battery or voltage to reproduce the same deflection on the galvanometer when a known resistance is placed in series with it.

The details of the commutator will be found described in other books by the author, so that it is unnecessary to repeat them here.[1] Suffice it to say that the arrangements are such

[1] See J. A. Fleming, "A Handbook for the Electrical Laboratory and Testing Room," Vol. II., p. 202, *The Electrician* Printing and Publishing Company, Ld., 1, Salisbury Court, Fleet Street, London, also "The Principles of Electric Wave Telegraphy and Telephony," 2nd Ed., p. 170, and "An Elementary Manual of Radiotelegraphy and Radiotelephony," p. 279, both the latter published by Messrs. Longmans, Green & Co., 39, Paternoster Row, London.

that the cable or capacity to be determined is charged and discharged a known number of times per second through a galvanometer by a known voltage.

One terminal of the galvanometer and one of the battery are connected together and to the earth or to one of the twin conductors or the outside sheath of the cable to be tested, and the other conductor is connected to the middle terminal of the commutator, the remaining battery and galvanometer connection being made to the two outer terminals of the commutator.

If there are N commutations per second and if the charging voltage is V and the capacity is C microfarads, then the current through the galvanometer is $NCV/10^6$. If this same deflection is restored when the voltage V is applied to the galvanometer through a resistance R which includes that of the galvanometer itself, then we must have

$$\frac{NCV}{10^6}=\frac{V}{R}, \text{ or } C=\frac{10^6}{RN}.$$

Hence the capacity is measured in microfarads by the reciprocal of the product of the total resistance in megohms and the frequency or number of discharges per second.

This method has the advantage that by employing a commutator running at a suitable speed we can determine the capacity corresponding to any required frequency within limits.

The method, however, does not separate out the true dielectric current from any conduction current unless certain precautions are taken. It is always desirable to make two sets of measurements, one with the galvanometer arranged so as to measure the series of charges given to the condenser and one in which it is arranged to measure the discharge current. If these two sets of measurements give different results the condenser has leakage as well as capacity.

Certain types of gutta-percha-covered wire or cable are known to be characterised by considerable true leakance as well as capacity. That is, the gutta-percha as a dielectric has a true conductivity, perhaps owing to moisture present in it, as well as dielectric quality. Hence many of the methods proposed for measuring capacity do not give correct results in the case of gutta-percha-covered wire or cable.

By any of the ordinary methods of measuring capacity it is difficult, if not impossible, to separate out the true conduction current from the true dielectric current. They can, however, be distinguished as follows:

If an alternating current is employed to send a current through

FIG. 3.—General view of Dr. Sumpner's Wattmeter.

a condenser the part of that current which depends upon capacity is expressed by $C\frac{dv}{dt}$, and if the potential difference of the plates, viz. v, is a simple sine function of the time of the form $v = V \operatorname{Sin} pt$, then the capacity current is measured by $CpV \operatorname{Cos} pt$, and is in quadrature as regards phase with the potential difference. If, however, the condenser possesses any true conductivity S, then the conduction current is Sv or $SV \operatorname{Sin} pt$, and this current is in step with the condenser potential difference.

Accordingly we can separate out these two components by any method which takes account only of the component in quadrature with the potential difference.

This is achieved by the use of Dr. Sumpner's iron-cored wattmeter.[1] This wattmeter, the general appearance of which is shown in Fig. 3, consists of a specially shaped laminated iron electromagnet (I) as in Fig. 4, wound over with a very thick copper wire. If this winding is connected to an alternating current circuit the impressed electromotive force is almost wholly expended in overcoming the reactance of the circuit, since the resistance is negligible. Accordingly if the instantaneous value of this impressed voltage is v, and if the

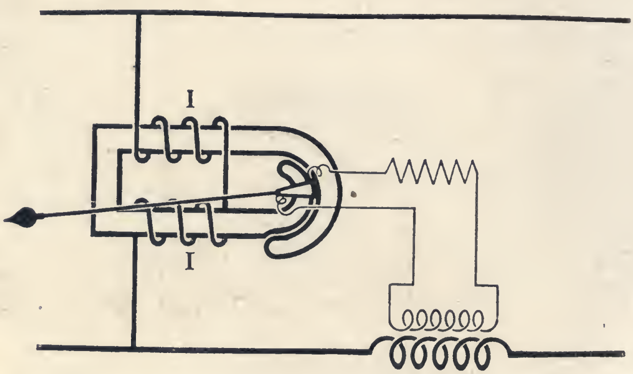

FIG. 4.—Arrangement of Circuits in Dr. Sumpner's Wattmeter.

corresponding total flux in the air gap of the electromagnet is represented by b, then, in accordance with Faraday's law, we have

$$v = -N\frac{db}{dt},$$

where N is the number of windings on the core of the electromagnet.

If then v varies in accordance with a simple sine law the magnetic flux must differ 90° in phase with it. In the narrow gap of this electromagnet a coil of wire can swing, and when a current i passes through this wire a force the mean value of

[1] See Dr. W. E. Sumpner, "New Alternate Current Instruments," *Jour. Inst. Elec. Eng.*, Vol. XLI., p. 237, 1908.

which is ib is excited causing the coil to move across the lines of flux. This is resisted by the torsion of a spring, and hence the deflection of the coil becomes a measure of the mean value of the product of the magnetic flux in the gap and the current i in the coil. Suppose then that this current is the current through a condenser which is placed in series with the coil and connected across the same terminals which supply the alternating voltage v. The current through this condenser, supposed to have leakance, consists, as above shown, of a component in step with the voltage and a component in quadrature with it. But this latter is in step with the magnetic field of the electromagnet,

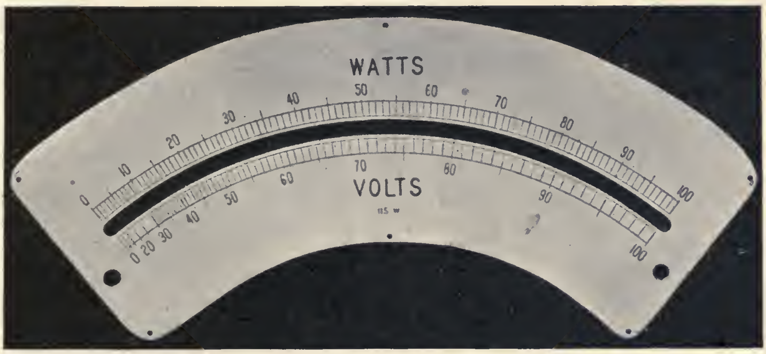

FIG. 5.—Scale of Dr. Sumpner's Wattmeter.

and the former is in quadrature with the field as regards phase. Accordingly it is only the true capacity current which contributes to deflect the coil, as that alone is in step with the magnetic field. The deflection of the coil is proportional to the mean product of ib, and therefore, if the scale over which the indicating needle moves is graduated, as shown in Fig. 5, to give the value of this product by inspection, we can obtain from the scale deflections the ratio between the known true capacity of a condenser which is placed in series with the coil and the true capacity of any other condenser or cable substituted for it, and dielectric leakage causes no error in this measurement.

This method is in extensive use for measuring the capacity of condensers for telephone work. For additional information on the measurement of the capacity of cables the reader is referred to the author's "Handbook for the Electrical Laboratory and Testing Room," Vol. II., p. 145, and to a paper by Mr. J. Elton Young on "Capacity Measurements of Long Submarine Cables," *Jour. Inst. Elec. Eng. Lond.*, Vol. XXVIII., p. 475, 1899.

7. The Practical Measurement of Inductance.—We shall also not attempt to mention all the various methods which have been suggested for the measurement of inductance, but confine ourselves to the consideration of one or two methods suitable for the determination of the inductance of cables with such frequencies as are used in telephony.

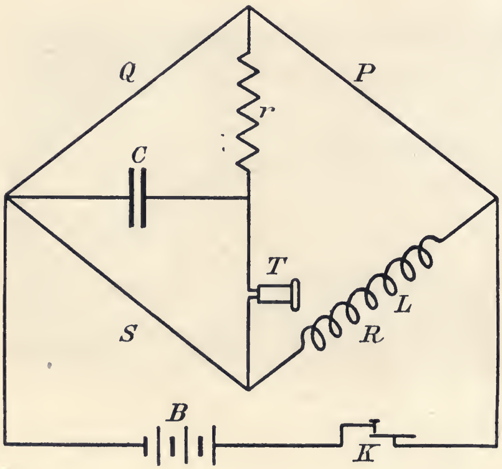

FIG. 6.—Anderson-Fleming method of measuring small inductances.

The author's experience has shown that one of the best of these is the method devised by Professor Anderson as modified by the author.

In this method the conductor R, L of which the inductance L is to be measured is inserted in one arm of a Wheatstone's bridge (see Fig. 6). If, for instance, we have to determine the inductance of a twin cable, it can be short-circuited at the far end and the two home ends joined into the bridge arm. If it is a single wire, such as an overhead telephone wire, then a loop of some kind must be formed enclosing a sufficiently large area so that the inductance is practically equal to that of a straight wire with the return far removed. The same applies to an armoured cable like a submarine cable. We cannot properly determine the inductance of such a single wire or cable when coiled in a tank or in a ship, because then the inductance of the cable is increased by the mutual inductance of the various coils or turns.

In any case, the conductor having been joined into the bridge, the bridge circuits, P, Q, and S are balanced in the usual way. The galvanometer must then have placed in series with it an adjustable resistance r and a condenser C arranged as in Fig. 6. The battery circuit must have a buzzer, or interrupter, K, placed in it so as to interrupt the battery current several hundred times per second. In place of the galvanometer a telephone T is inserted. The bridge arms having been adjusted to obtain a steady balance, so that no current flows through the galvanometer when the buzzer is short-circuited, we switch over to the telephone and replace the buzzer. A loud sound will then be heard in the telephone, and this must be annulled by inserting resistance r in series with the telephone. When silence has been obtained the inductance L of the cable under test is given by the formula below.

Let the four resistances forming the arms of the bridge be P, Q, R, S, R being the resistance of that arm which includes the inductance L. Let x be the current in arm Q, and let z be the current in the resistance r and y that in the inductive resistance LR.

If then the bridge is balanced so that $P : Q = R : S$ there will be no current in the galvanometer when the battery current is steady. If r is so adjusted that there is no current in the telephone when the battery current is interrupted, then the fall of potential down S must be equal to the fall of potential down Q and r, and the current in r must be the same as the condenser current. Also the fall of potential down P must be the same as that down the inductive resistance LR. These conditions expressed in symbols are

$$Qx = Sy + rz,$$

$$rz + P\,(x+z) = Ry + L\,\frac{dy}{dt},$$

$$PS = QR, \text{ and } \frac{1}{C}\int z\,dt = Sy.$$

From these equations we easily find that

$$\left\{ r + P + \frac{P}{Q}r \right\} z = L\,\frac{dy}{dt} = \frac{L}{CS}\,z.$$

Hence $$L = C\{S(r+P)+Rr\},$$

or $$L = C\{r(R+S)+RQ\} \quad . \quad . \quad . \quad . \quad (41)$$

In measuring small inductances the capacity C should be small. The method is sufficiently sensitive to measure the inductance of a few yards of wire provided that the value of C is accurately known. If the inductive resistance has iron involved in its construction, then the inductance will vary with the current through it unless that current is either very large or very small. For the purposes of this test it is a great convenience to have a small alternator giving an electromotive force which can be varied by the excitation and a frequency which is between 500 and 1,000. We can then determine the inductance for telephonic frequencies.

8. The Measurement of Small Alternating and Direct Currents.—The small alternating or periodic currents with which we are concerned in telephony are best measured by means of some form of thermoelectric ammeter. The ordinary telephonic current is a current of a few milliamperes created by an electromotive force of 2 to 10 volts, and is of complex wave form.

According to Mr. B. S. Cohen, the frequency of the fundamental harmonic lies generally between 100 and 300, and that of the highest harmonic between 4,000 and 5,000, although harmonics above 1,500 are comparatively unimportant.[1]

The average frequency of the telephone speech current is about 800. Hence for currents of such frequency almost the only reliable method of current measurement is by some form of thermal ammeter.

Mr. Duddell has devised a very sensitive thermoelectric ammeter with negligible inductance. The current to be measured is passed through a small wire or metallic strip, which may be gold-leaf, supported on a non-conducting base. Over this strip is suspended by a quartz fibre a light bismuth-antimony thermocouple, one junction of which nearly touches the wire or strip.

[1] See Mr. B. S. Cohen, "On the Production of Small Variable Frequency Alternating Currents suitable for Telephonic and other Measurements," *Phil. Mag.*, September, 1908, also *Proc. Phys. Soc. Lond.*, Vol. XXI.

This thermocouple hangs in a strong magnetic field, and when a current is passed through the strip it is heated; this heats the thermojunction by radiation and convection, and the current so created causes the thermocouple, which is in the form of a long narrow loop, to be deflected. The deflection is rendered visible by a light mirror attached to the thermocouple, from which a ray of light is reflected to a scale. A general view of the instrument is shown in Fig. 7. It can be calibrated

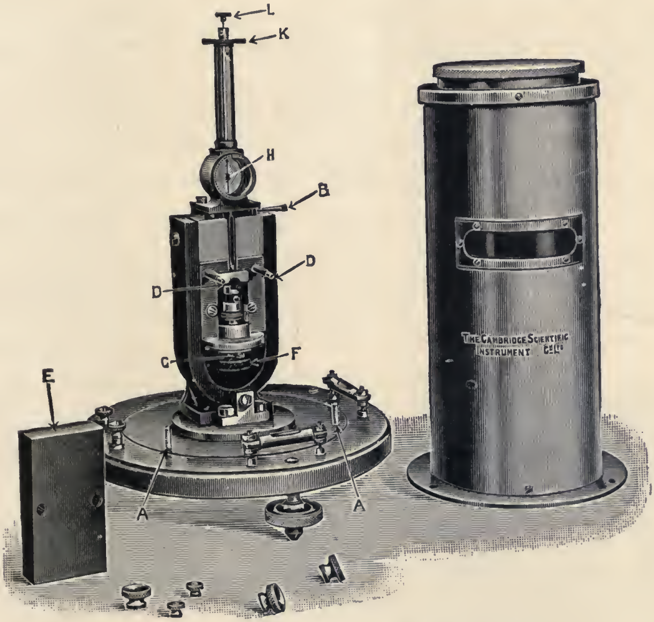

FIG. 7.—Duddell's Thermogalvanometer.

by passing known small continuous currents through the heated strip. To secure good readings the instrument must be placed on a very steady support free from every trace of vibration. It is, however, a very suitable instrument for the measurement of the root-mean-square (*R.M.S.*) values of such currents as are usual in telegraph and telephone cables. By the employment of suitable heater resistances it can be used for large alternating currents.

Another useful current-measuring instrument is the barretter of Mr. B. S. Cohen. The sensitive portion consists of a pair of small carbon filament 24-volt glow-lamps. When the carbon filament is heated the resistance decreases. The two glow-lamps are joined up as shown in Fig. 8. Each glow-lamp, called in this

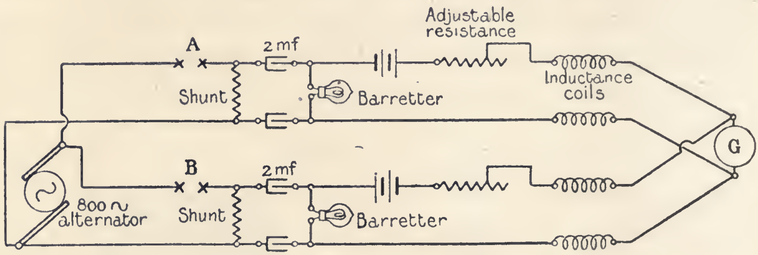

FIG. 8.—Arrangement of Circuits in Cohen's Barretter.

connection a barretter, has a pair of 2-mfd. condensers attached to its terminals and a shunt connecting them. On the other side a few cells of a storage battery and an adjustable resistance and inductance coil are connected as shown in the diagram. The batteries can send current through the carbon filaments, but not through the condensers, whilst, on the other hand, alternating currents can pass through the condensers, but are throttled by the inductance coils. In each alternating current branch of each circuit there is an interruption, marked A and B respectively.

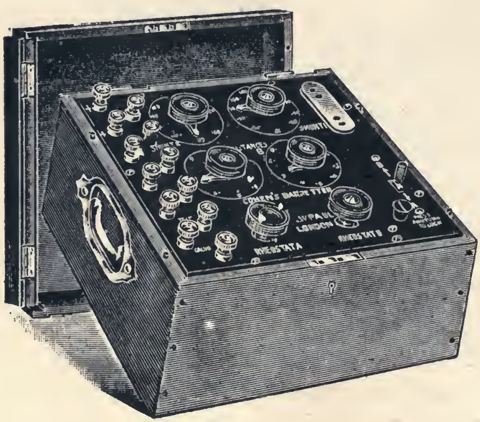

FIG. 9.—General appearance of the Cohen Barretter as made by Mr. R. Paul.

In using the instrument the adjustable resistances are given such values that the continuous currents balance one another, and the galvanometer, G, remains at zero. Suppose then the alternator removed, and that some circuit in which there is a feeble alternating current is connected on at one gap, A. This alternating current flows partly through one barretter and lowers the resistance of the filament, and, the balance being upset, the galvanometer deflects. The instrument may be calibrated by sending through it various small alternating currents, which pass also through a known inductionless resistance. The drop in potential down this resistance can be measured by an electrostatic voltmeter, also previously standardised, and the measured fall in potential gives the value of the alternating current, which can then be compared with the observed deflection of the galvanometer. The process of calibration is more difficult than in the case of a simple thermal ammeter, but when once carried out the barretter can be used to determine the ratio of the currents at two distant points in a telephone cable, and hence the attenuation constant of the cable. The general appearance of the barretter is as shown in Fig. 9.

9. The Measurement of Small Alternating Voltages. The Alternate Current Potentiometer.—When the voltage to be measured is not very small it can be conveniently determined by a Dolezalek electrometer, which consists of a quadrant electrometer of the Kelvin pattern but having a "needle" made of silver paper suspended by a quartz fibre. The instrument is used as an idiostatic electrometer by connecting the needle to one of the quadrants. If, however, the voltage in question amounts only to a few volts or fractions of a volt, an idiostatic quadrant electrometer will hardly be sufficiently sensitive. Recourse may then be had to an alternating current potentiometer, such as the Drysdale-Tinsley form, which is admirably suited for many of the measurements to be made in connection with cables. This last instrument consists of a standard form of potentiometer as used for direct current work, but it is supplemented by means for passing through the standard wire an alternating current of known value derived from the same source as the potential to be measured, and also with means for shifting the phase of this current and changing its amplitude.

The phase shifting is accomplished by one of Dr. Drysdale's phase-shifting transformers (see Fig. 10). If a laminated iron ring is wound over in four quadrants with coils connected pair and pair, and if these two pairs are joined into the two sides of a two-phase alternator giving two simple harmonic voltages differing 90° in phase, we can produce thereby a rotating magnetic field in the interior space. If in this space is placed a core wound over with one winding in one plane, then if this winding is placed with its plane perpendicular to the field of one pair of coils on the stator, an *E.M.F.* will be induced in it, and if the coil is turned so as to be perpendicular to the other stator field it will have an *E.M.F.* differing 90° in phase from the former induced in it. By turning this secondary coil into any intermediate position it will have an *E.M.F.* induced in it which has the same amplitude but with intermediate phase, and shifted proportionately to the angle through which it is turned. We can obtain the two stator currents in quadrature from one single-phase alternator by introducing a shunted condenser into one circuit, as shown in Fig. 11.

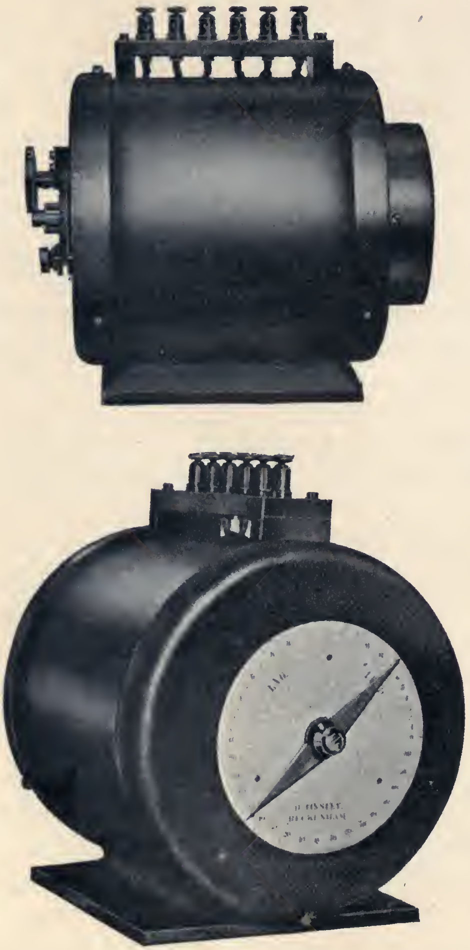

Fig. 10.—Drysdale Phase Shifting Transformer as made by Mr. H. Tinsley.

Hence the phase-shifting transformer can be made up as one self-contained appliance workable off any constant single-phase circuit giving a simple sine curve *E.M.F.*[1]

Returning then to the Drysdale-Tinsley potentiometer, we give in Fig. 12 a perspective view of the instrument and in Fig. 13 a diagram of the connections.[2] The instrument consists of a standard form of direct current Tinsley's potentiometer, to which is added an electrodynamometer or mil-ampere meter for indicating the current in its slide wire. A phase-shifting transformer can have its secondary circuit put in series with this wire by a throw-over switch. Then, when using an alternating current, the ordinary movable coil galvanometer is replaced by a vibration galvanometer in which the needle is a small piece of soft iron suspended by a wire in the field of a strong magnet, which can be varied by a magnetic shunt (see Fig. 14). A coil behind the iron carries the alternating current. When an alternating current passes through this coil the needle is set in vibration, and if the magnetic field is varied so that the natural time period of the vibrating needle is the same as that of the alternating current, the amplitude of motion becomes very large, and is observed by throwing a ray of light upon a mirror attached to the needle. Means are provided for varying by rheostats the current in the slide wire of the potentiometer. If, therefore, we desire to know the value as regards magnitude and phase of the alternating potential

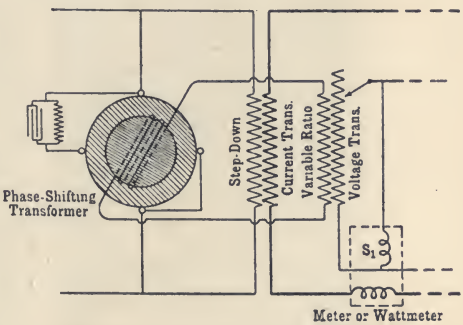

FIG. 11.—Diagram showing the manner in which two currents in phase quadrature can be obtained from a single phase current by means of a shunted condenser.

[1] See Dr. C. V. Drysdale, "The Use of a Phase-shifting Transformer for Wattmeter and Supply Meter Testing," *The Electrician*, Dec. 11th, Vol. LXII., p. 341, 1908.

[2] See Dr. C. V. Drysdale, "The Use of the Potentiometer on Alternate Current Circuits," *Phil. Mag.*, March, Vol. XVII., p. 402, 1909, or *Proc. Phys. Soc. Lond.*, Vol. XXI., p. 561, 1909.

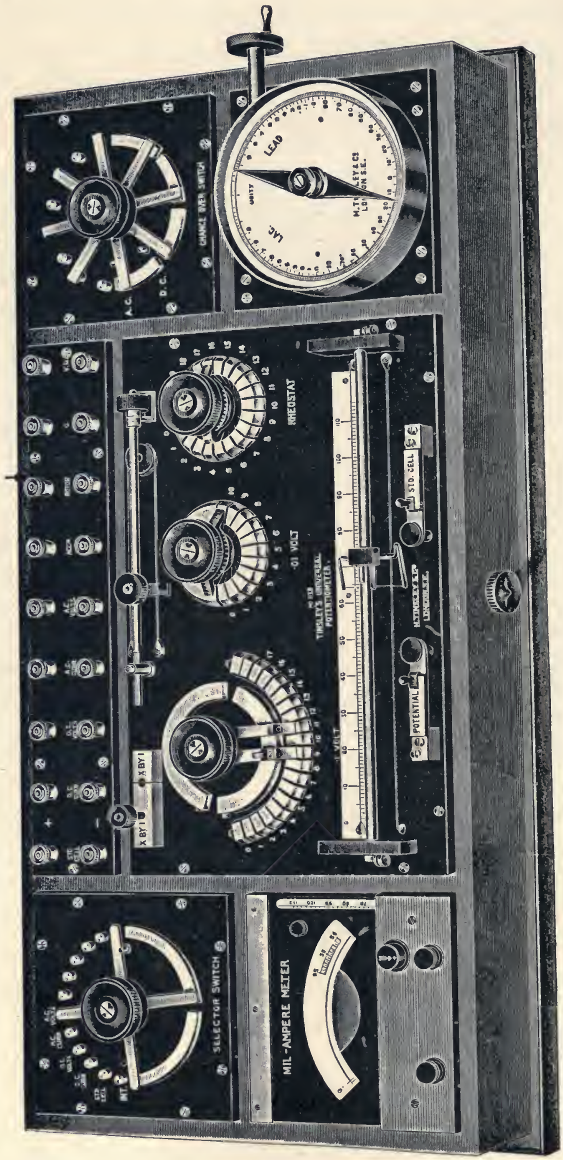

FIG. 12.—General view of the Drysdale-Tinsley Alternating-Direct Current Potentiometer.

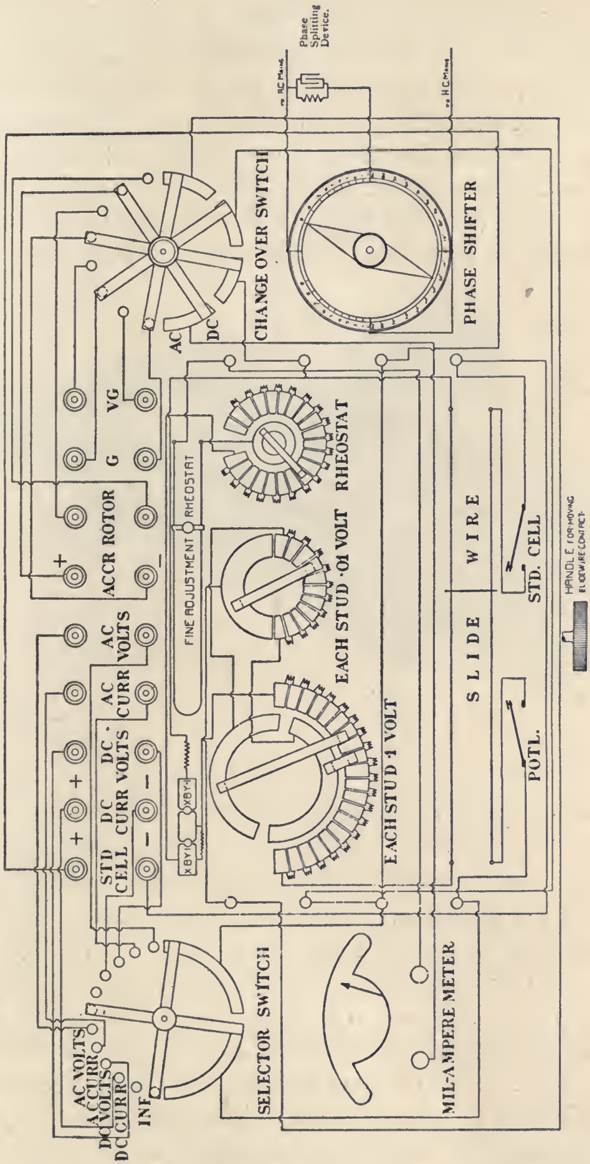

FIG. 13.—Scheme of Connections of the Drysdale-Tinsley Alternating-Direct Current Potentiometer.

difference between two points or between the ends of a non-inductive resistance carrying an alternating current, we bring from these points two wires to the potentiometer in the usual way, and balance this unknown alternating potential difference (*A.P.D.*) against the fall of potential (also alternating) down the slide wire, and adjust the strength and phase of this fall by the rheostats and phase shifter until the vibration galvanometer shows no current (see Fig. 15). To do this the current in the slide wire must be provided from the same source as that which

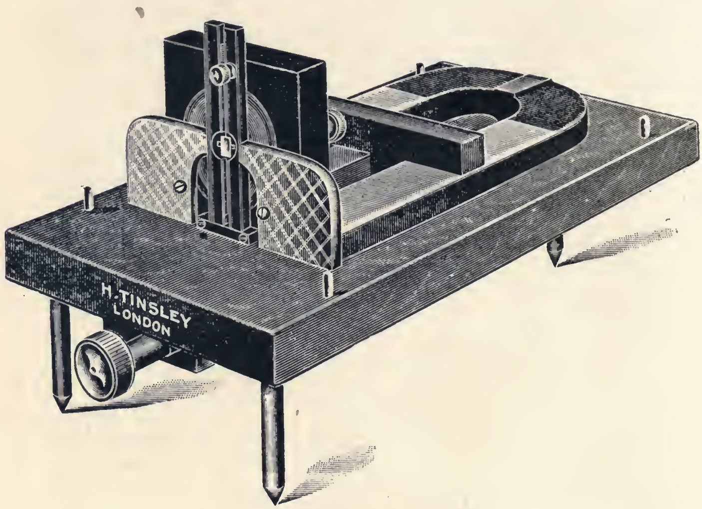

Fig. 14.—Tinsley Vibration Galvanometer for use with A. C. Potentiometer.

supplies the current or potential difference under test, so that the frequency is the same. The phase of the *A.P.D.* under test is then read off at once on the dial of the phase-shifting transformer, which is shown at the right-hand bottom corner in Figs. 12 and 13. We have to balance the *A.P.D.* to be tested against the known *A.P.D.* between two points on a slide wire in which is a current of known value, the phase of which can be shifted if need be through 360°. The current in this wire is kept at a known value and equal to that of a standard direct current, which last can be adjusted by a standard Weston cell in the usual way.

The instrument forms therefore a valuable means of measuring small alternating currents both for strength and phase difference. We can by means of it determine the current and phase of that

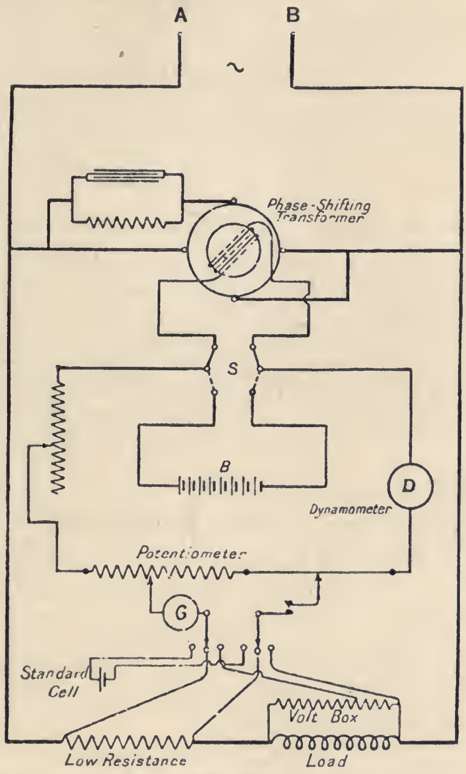

FIG. 15.—Scheme of Connections used in making tests with the Drysdale-Tinsley A. C. Potentiometer. The points A, B are the terminals of a 100-volt alternator or transformer.

current at any point in a long cable to which an alternating electromotive force is applied.

10. The Measurement of Attenuation Constants of Cables.—If the current at any point in a cable is I_1 and that at any other point separated by a distance l is I_2, and if a is the attenuation constant of the cable, then the equation which connects the above quantities is

$$(I_2) = (I_1)\,\epsilon^{-al} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where (I_1) and (I_2) signify the strengths of these currents without regard to phase difference.

Hence $$\frac{(I_1)}{(I_2)} = \epsilon^{al} \text{ and } a = \frac{1}{l} \log_\epsilon \frac{(I_1)}{(I_2)} \quad . \quad . \quad . \quad (43)$$

or, using ordinary logarithms,

$$a = \frac{1}{l}\, 2{\cdot}3026 \log_{10} \frac{(I_1)}{(I_2)} \quad . \quad . \quad . \quad . \quad (44)$$

The attenuation constant a is therefore quite easily measured by inserting in the run of the cable at two points separated by a known distance l two hot wire ammeters or two barretters which agree absolutely together and measuring with them the *R.M.S.* value of the currents in the cable at the two places. The attenuation constant is the Napierian logarithm of the ratio of these currents divided by the distance in miles or nauts.

11. Measurement of the Wave Length Constant of a Cable.—The wave length constant β of a cable is defined to be an angle β in circular measure such that the phase difference in the currents at two points in the cable separated by a distance l is βl. Accordingly it can be measured by means of a Drysdale-Tinsley alternate current potentiometer or by any other means which enables us to measure the phase difference between the currents.

12. Measurement of the Propagation Constant of a Cable.—The propagation constant P of a cable is defined by the equation $P = a + j\beta$, where a is the attenuation constant and β is the wave length constant. Accordingly P is known when a and β are separately determined. It is, however, best measured by determining the final sending end impedance with far end open and closed as shown in the next section.

13. Measurement of the Initial Sending End Impedance of a Cable.—We have defined the initial sending end impedance Z_0 of a cable in Chapter III., § 4, as the quantity

$$Z_0=\frac{\sqrt{R+jpL}}{\sqrt{S+jpC}} \quad . \quad . \quad . \quad . \quad (45)$$

It is a vector quantity and is measured in vector ohms and expressed in the form $(X)\underline{/\theta}$, where (X) is some number of ohms and θ is some phase angle.

We have also seen that the *final sending end impedance* Z_1 is defined by the equation

$$Z_1=\frac{V_1}{I_1}$$

where V_1 is the simple periodic electromotive force applied to the sending end of a cable and I_1 is the current flowing into it at the sending end.

Suppose that the ratio V_1/I_1 is measured when the far end of the cable is open or insulated and call the value Z_f, then we have seen (Chapter III.) that

$$Z_f=Z_0 \operatorname{Coth} Pl \quad . \quad . \quad . \quad . \quad (46)$$

Again, if the final sending end impedance is measured with the far end of the cable short circuited, and if we call this value Z_c, we have seen that

$$Z_c=Z_0 \operatorname{Tanh} Pl \quad . \quad . \quad . \quad . \quad (47)$$

Hence multiplying together the equations (46) and (47) we have

$$Z_0=\sqrt{Z_f Z_c}. \quad . \quad . \quad . \quad . \quad (48)$$

The process of measuring the initial sending end impedance consists therefore in measuring the ratio of the applied voltage V_1 to the current at the sending end when the receiving end is insulated and when it is short-circuited. It must be remembered that V_1 and I_1 in both cases are quantities differing in phase as well as magnitude. Hence their ratio is a vector, and therefore the geometric mean $\sqrt{Z_f Z_c}$ is a vector and is expressed in vector ohms.

The measurement can be made either with a Drysdale-Tinsley potentiometer or with a Cohen barretter. It involves measuring the value of I_1 in the two cases and the difference in phase of this current and the impressed voltage V_1 in the two cases, but it is the best means of measuring the initial sending end impedance Z_0 which appears in so many of the formulæ. This method of measurement enables us also to calculate the value of $S + jpC$ for any cable, as the values of S and C are less easy to measure experimentally than those of R and L.

Since $Z_c = Z_0 \tanh Pl$
and since $Z_0 = \sqrt{Z_f Z_c}$ it follows that

$$\text{Tanh } Pl = \sqrt{\frac{Z_c}{Z_f}} \quad . \quad . \quad . \quad . \quad (49)$$

and therefore that

$$P = \frac{1}{l} \tanh^{-1} \sqrt{\frac{Z_c}{Z_f}} \quad . \quad . \quad . \quad . \quad (50)$$

This gives the best means of determining the propagation constant experimentally in the case of any given cable. Since P is an abbreviation for the product $\sqrt{R + jpL}\,\sqrt{S + jpC}$ and Z_0 stands for the quotient $\sqrt{R + jpL}/\sqrt{S + jpC}$ it follows that

$$PZ_0 = R + jpL$$

$$\frac{P}{Z_0} = S + jpC.$$

Hence substituting the values of P and Z_0 given above we have

$$R + jpL = \frac{\sqrt{Z_f Z_c}}{l} \tanh^{-1} \sqrt{\frac{Z_c}{Z_f}} \quad . \quad . \quad . \quad (51)$$

$$S + jpC = \frac{1}{\sqrt{Z_f Z_c}\, l} \tanh^{-1} \sqrt{\frac{Z_c}{Z_f}} \quad . \quad . \quad . \quad (52)$$

The experimental determination therefore of Z_f and Z_c leads at once to a knowledge of the vector impedance $R + jpL$ and the vector admittance $S + jpC$.

14. Measurement of the Impedance of various Receiving Instruments.—The measurement of the inductance effective resistance and vector impedance of various types of receiving instrument is an extremely important matter because no predeterminations can be made of the current at the receiving end of a line unless we know the impedance of the receiving instrument. Some very valuable measurements of this kind have been carried out by Mr. B. S. Cohen in the investigation laboratory of the National Telephone Company and are recorded in the *National Telephone Journal*[1] for September, 1909, by methods described lower down. Also other methods of measurement have been elaborated by Messrs. B. S. Cohen and G. M. Shepherd which are described in a paper on Telephonic Transmission Measurements read before the Institution of Electrical Engineers of London in 1907,[2] in which the Cohen barretter is employed. This instrument has already been described in principle in § 8 of this chapter.

By it the following measurements can easily be made:

1. The impedance of any piece of telephonic apparatus expressed in ohms for any type of alternating current.

2. By employing an alternator giving a simple periodic or sine form *E.M.F.* the actual inductance and effective resistance and capacity of any piece of apparatus for these high frequency currents can be obtained.

3. Small alternating currents can be measured with an ordinary galvanometer.

4. The direct comparison of various types of cables with the performance of a standard cable can be made.

The barretter can be used with modification to measure the impedance of any piece of telephonic apparatus. For this purpose a source of electromotive force must be provided having approximately a simple sine wave form, and a frequency of about 800. Also the shunt (see Fig. 8) must be replaced by a telephone induction coil and a large condenser (10 mfd.) placed across the galvanometer terminals.

Many forms of alternator have been devised for this purpose, some of which are described in the author's work, "Principles of Electric Wave Telegraphy and Telephony," Chap. I.

The Western Electric Company of America supply a machine having an output of about 30 watts at frequencies varying from 800 to 1,800, and the wave form is stated to resemble a sine curve closely at all loads.

Messrs. Siemens and Halske also make a machine with an output of 3 or 4 watts with the same frequencies. This machine

[1] Published at Telephone House, Victoria Embankment, London.
[2] See *Journal of Proc. Inst. Elec. Eng. Lond.*, Vol. XXXIX., p. 503, 1907.

is of the inductor type, and the purity of the wave form is preserved by appropriately shaping the teeth.

The investigation department of the National Telephone Company constructed a small inductor machine giving a small output but approximately sine form of wave.

For accurate measurements this machine can be supplemented by a wave filter consisting of a series of inductance coils of low resistance with condensers parallelised across, and this circuit is so designed as to obstruct the passage of harmonics and preserve the fundamental sine term in the wave form.

Such a wave filter was described by Mr. G. A. Campbell in an article in the *Philosophical Magazine* for March, 1903.[1]

A fairly good test of the simple sine form of the *E.M.F.* of an alternator is to employ it to charge some form of condenser and measure the charging current. If this agrees with that calculated from the expression $A = \frac{pCV}{10^6}$ where C is the capacity in microfarads, V the *P.D.* of the condenser terminals in volts, and A the charging current in amperes, then the *E.M.F.* wave form is very probably a pure sine curve.

Returning then to the actual measurement of the impedance of some form of telephonic apparatus, let R_0 be the effective resistance of the apparatus. This must not be confused with the true steady or ohmic resistance. It is much greater, first, because the *H.F.* current in the conductor is not uniformly distributed over the cross section of the wire; secondly, because the current in neighbouring turns of wire furthermore increases this non-uniformity; and thirdly, because the dissipation of energy in any iron core which may be present in the form of eddy currents or magnetic hysteresis loss is a dissipation of energy which counts as if due to an increase in the actual resistance.

In the next place the apparatus has inductance L_0, and at a frequency n when $n = p/2\pi$ we have an impedance $\sqrt{R_0^2 + p^2L_0^2}$ in the apparatus.

Suppose then the telephonic apparatus under test is inserted

[1] See also Mr. B. S. Cohen, "On the Production of Small Variable Frequency Alternating Currents," *Phil. Mag.*, September, 1908, or *Proc. Phys. Soc. Lond.*, Vol. XXI., p. 283, 1909.

in one gap B in the Cohen barretter circuits (see Fig. 8) and a variable inductionless resistance is inserted in the other gap A, and let a high frequency sine wave alternator be connected in as shown in the diagram.

Let the barretter or glow lamp and shunt across its terminals together with the condensers in series (2 mfds.) have an equivalent resistance r. The first step is to balance on the galvanometer any inequality in the electromotive force of the two batteries inserted in front of the barretters. This is done by the adjustable resistances. The alternator is then started and the variable inductionless resistance R_1 in the gap A is altered until it balances the effect of the impedance $\sqrt{R_0^2 + p^2L_0^2}$ in the gap B, and the galvanometer then shows no current because the effective impedance in the circuits of both barretters is the same.

If then the resistance of each of the shunts in the barretter across the pair of condensers in each side is denoted by r, and since the $E.M.F.$ in two circuits is the same and the currents the same, we have an equality between the total resistances or resistance and impedances, or in other words the equation

$$\sqrt{(R_0+r)^2+p^2L_0^2}=R_1+r \quad . \quad . \quad . \quad (53)$$

We need not take into account the reduction in the shunt resistance r which results from it being shunted by a galvanometer provided the latter has, as it should have, a resistance of several thousand ohms.

Squaring both sides of (53) we have

$$(R_0+r)^2+p^2L_0^2=(R_1+r)^2 \quad . \quad . \quad . \quad (54)$$

Hence $$R_0^2+p^2L_0^2=R_1^2+2r\ (R_1-R_0)$$

or $$\sqrt{R_0^2+p^2L_0^2}=\sqrt{R_1^2+2r\ (R_1-R_0)} \quad . \quad . \quad (55)$$

This gives us the impedance of the instrument.

To separate out the effective resistance R_0 from the reactance we may proceed as follows: Add in series with the telephonic apparatus an inductionless resistance r_1 and proceed as before to obtain a balance against an inductionless resistance of value R_2 in the other side of the barretter. Then we have the equation

$$(R_0+r_1+r)^2+p^2L_0^2=(R_2+r)^2 \quad . \quad . \quad . \quad (56)$$

and since by (54) we have

$$(R_0+r)^2+p^2L_0^2=(R_1+r)^2 \quad . \quad . \quad . \quad (57)$$

we have two simultaneous equations to determine pL and R.

Hence $$R_0 = \frac{R_2^2 - R_1^2 + 2r(R_2^2 - R_1) - 2rr_1 - r_1^2}{2r_1} \quad . \quad . \quad (58)$$

$$L_0 p = \sqrt{(R_1 + r)^2 - (R_0 + r)^2} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

From which we obtain $\tan\theta = \frac{pL_0}{R_0}$, θ being the phase angle of the vector impedance $\sqrt{R_0^2 + p^2 L_0^2}$.

Mr. Cohen finds that the above method of measuring the effective resistance and inductance of telephonic apparatus can give good results provided that the shunts shown in Fig. 8

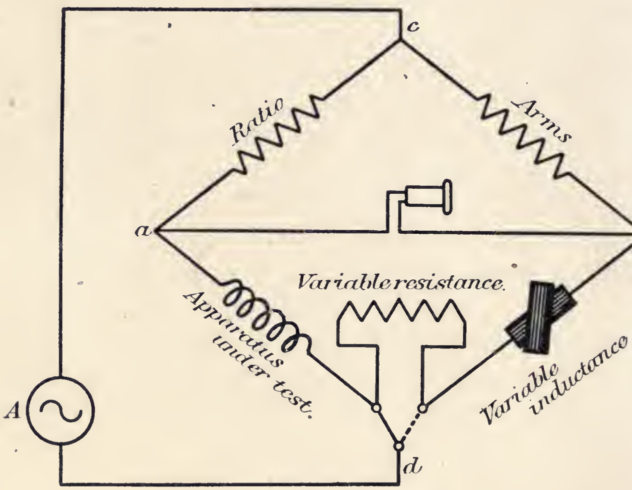

FIG. 16.—Arrangement of Circuits for measuring the vector-impedance of any telephonic apparatus.

across the barretter circuits are replaced by telephone induction coils separating the alternator and gaps A and B from the barretter circuits, and also that a condenser of large capacity is placed across the galvanometer terminals.

Another method of making these measurements which requires no special instrument not usually found in the laboratory except the high frequency alternator was adopted by Mr. B. S. Cohen in making the measurements of instruments given below. In this arrangement the alternator is applied to the battery terminals of a Wheatstone's Bridge (see Fig. 16) and in the bridge circuit is placed a telephone receiver. The instrument to be tested is placed in one arm of the bridge, and in the adjacent arm is inserted a variable inductionless resistance and a low resistance variable inductance. These are independently adjusted to give silence in the telephone and enable the effective resistance R_0 and inductance L_0 to be separately equilibrated by resistance plugged out of the box and inductance inserted in the arm. This inductionless resistance is made on a plan suggested by Mr. Duddell. The resistance material is a kind of cloth woven with a silk warp and fine resistance wire woof and has the property of possessing extremely small inductance and capacity, which is more than can be said for the ordinary plug resistance boxes of most laboratories. The inductance is made with two coils, one outside the other, the inner one capable of rotating on an axis so as to be turned in such positions as to vary the mutual inductance of the two parts and therefore the self inductance of the two in series. Turning then to the results obtained by Mr. Cohen, we give on p. 228 a table published by him in the *National Telephone Journal* for September, 1909.

The figures in the fourth and fifth columns give respectively the scalar impedance in ohms and the vectorial angle $\tan^{-1}\frac{pL}{R}$ of the instrument.

It will be seen that the effective resistance is always much greater than the ohmic or steady resistance. Thus a so-called 60 ohm Bell telephone receiver has an effective resistance of 134 ohms, an inductance of 18 millihenrys, an impedance of 176 ohms, and the angle of lag of current behind terminal *P.D.* is 40° 24′.

The last column gives the power absorption of the instrument in milliwatts per volt *P.D.* at the terminals, and the total power loss is obtained by multiplying these numbers by the square of the terminal potential difference in volts.

We thus have determined for us the value of the Z_r which appears in many formulæ in Chapter III. as the vector impedance of the terminal instruments.

TABLE GIVING THE EFFECTIVE RESISTANCE R_0 AND INDUCTANCE L_0 AND IMPEDANCE FOR VARIOUS TELEPHONIC INSTRUMENTS AT A FREQUENCY $n = 1000$ OR $p = 6280$. (MR. B. S. COHEN.)

Apparatus.	S.L. No.	Effective resistance. Ohms.	Inductance. Henrys.	Impedance.		Loss in milliwatts per 1 volt.
				Ohms.	Angle.	
Bells.						
1,000ω magneto . .	6	7,580	1·305	11,140	47° 9′	·061
Indicators.						
1,000ω tubular, ordinary	10	8,000	1·2	11,000	43° 24′	·066
Do. do. differential	11	20,200	·224	20,300	5° 0′	·049
600ω self-restoring	5	8,055	1·3	11,410	44° 55′	·062
100ω + 100ω eyeball signal, unoperated	—	3,900	0·512	4,035	14° 45′	·240
100ω + 100ω eyeball signal, operated	—	4,300	0·539	4,440	14° 3′	·219
Instruments.						
Local battery subscribers, battery key up	1	434	0·189	1,265	69° 57′	·027
Do. do. down	1	563	0·182	1,275	63° 48′	·035
Receivers.						
Double pole Bell (60ω central battery)	10	134	·0182	176	40° 24′	4·33
Relays.						
500ω double make and break. (W.E.) armature not attracted	9	7,160	1·157	10,210	44° 54′	·069
Do. do., attracted	9	7,960	1·238	11,150	44° 24′	·064
1,000ω do. do., not attracted	11	9,910	1·543	13,845	44° 18′	·052
Do. do., attracted	11	9,970	1·617	14,230	45° 30′	·049
Retards.						
100ω tubular . .	—	1,116	0·191	1,640	47° 6′	·414
200ω ,, . .	—	3,170	0·550	4,690	47° 30′	·144
400ω ,, . .	5	4,700	0·664	6,280	41° 30′	·119
600ω ,, . .	1	5,906	0·890	8,132	43° 20′	·089
1,000ω ,, differential	2	19,100	0·538	19,400	10° 0′	·051
75ω + 75ω W.E. pattern, No. 2020A	—	1,827	1·367	8,770	77° 58′	·024
200ω + 200ω W.E. toroidal, No. 44B	—	3,600	13·5	85,000	87° 34′	·0005
No. 1, Central Battery Termination (consisting of repeater, supervisory relay, local line and subscriber's instrument).						
(*a*) No. 25 repeater, local line, 0ω	—	330	0·049	451	42° 57′	1·62
(*b*) Do. do. 300ω (ohmic)	—	630	0·068	760	33° 54′	1·09
(*c*) Do. do. 3-m. 20-lb. cable	—	680	0·049	746	23° 51′	1·22

15. The Power Absorption of various Telephonic Instruments.—The measurement of the energy absorbed by telephonic apparatus under working conditions presents, as Messrs. Cohen and Shepherd remark, considerable difficulty.[1] This energy is extremely small, perhaps only a few microwatts, and is always a variable quantity. The difficulty is to find any instrument which when inserted in circuit with the instrument to be tested does not seriously alter the conditions of test.

Messrs. Cohen and Shepherd have made a number of such measurements, employing a method due to Mr. M. B. Field, as follows. If a small transformer of suitable design has one of its coils inserted in parallel with the instrument under test, and if a suitable inductionless resistance is inserted in series with the instrument, we can draw off from the secondary of the transformer a current proportional to the *P.D.* at the terminals of the instrument tested, and from the terminals of the inductionless resistance a current proportional to the current in that instrument. Let i be the current at any instant in the instrument tested and therefore in the inductionless resistance R in series with it. Then Ri is the voltage at the terminals of this resistance. Let v be the potential difference at the terminals of the instrument tested, then the *P.D.* at the terminals of the secondary circuit will be Gv where G is some constant.

A Duddell thermo-galvanometer having a heater with a resistance of 100 ohms was then arranged with switches so that either the sum or the difference of these two voltages could be applied to send a current through a thermo-galvanometer *T.G.*

Let D_1 and D_2 be the instantaneous values of the sum or differences of the above voltages, viz.,

$$D_1 = Ri + Gv$$

$$D_2 = Ri - Gv$$

Then $$\frac{D_1^2 - D_2^2}{4RG} = vi.$$

[1] See Messrs. Cohen and Shepherd on Telephonic Transmission Measurements, *Journal Inst. Elec. Eng. Lond.*, Vol. XXXIX., p. 521, 1907.

Hence if we take mean values throughout a period and denote these by $(D_1)^2$ $(D_2)^2$, (V), and (I) we have

$$\frac{(D_1)^2-(D_2)^2}{4RG}=(V)\,(I)\,\mathrm{Cos}\,\phi \quad . \quad . \quad . \quad (60)$$

where ϕ is the power factor. The right-hand side of the above equation is the mean value of the power taken up in the telephonic instrument and $(D_1)^2$ and $(D_2)^2$ will be proportional to the deflections in the two cases of the thermo-galvanometer.

The above formula presupposes that the non-inductive resistance R is very small compared with the resistance of the thermo-galvanometer.

The transformer used by Messrs. Cohen and Shepherd had a toroidal core of No. 40 *S.W.G.* iron wire 11·5 cm. outside diameter and 5 cm. deep, and a cross section of 7·89 cms. Its two windings had respectively 2,000 and 100 turns and a transformation ratio from 96·5 to 19·3 according to the number of secondary turns used.

The following results were obtained. In a test made with 30 miles of 20-lb. paper insulated telephone cable with far end open, the sending end impedance was found as follows:—At a frequency of 810 the current into the line was 0·00658 amp. The power absorbed by the line was 0·0163 watts, and the power factor was 0·71. Hence since the cable is fairly long this gives us the initial sending end impedance $Z_0 = 552$ ohms with phase angle 44° 48′ downwards or $Z_0 = 552\ \overline{\diagdown 44^\circ\, 48'}$.

This is in fair agreement with the calculation made from the four cable constants.

The reader should note that the same method can be employed to determine the final sending end impedance when the cable is open or short circuited at the receiving end. We have to measure, in that case, the current into the cable at the sending end I_1, the applied voltage or *E.M.F.* V_1, and the power taken up by the cable W.

The ratio $\frac{(V_1)}{(I_1)}$ or the ratio of the *R.M.S.* value of the voltage and current gives the numerical value or size of the impedance Z_1. Also the ratio of the true power taken up W in watts to the product of (V_1) and (I_1) or to the volt-amperes gives us Cos ϕ or the power factor. From which we have ϕ or the phase angle. Hence $\frac{(V_1)}{(I_1)} = (Z_1)$ and $\frac{W}{(V_1)\,(I_1)} = \text{Cos}\ \phi$ and the vector final sending end impedance $Z_1 = (Z_1) \underline{\lceil \phi}$.

In the same manner we can find Z_f, and Z_c, and therefore Z_0.

For various receiving instruments the following results were obtained by Messrs. Cohen and Shepherd.

Apparatus tested. Frequency 825.	Current in amperes.	Power in watts.	Power Factor.	Effective Resistance in ohms.	Inductance in henrys.
Central Battery Receiver	0·00695	0·00858	0·600	165	0·0425
120-ohm Receiver .	0·01160	0·02200	0·760	165	0·0280
120-ohm Receiver and Induction Coil	0·00220	0·00139	0·562	227	0·0650
Central Battery Repeater with 150-ohm Subscriber's Line	0·00208	0·00149	0·685	320	0·0690

16. Determination of the Fundamental Constants of a Cable from Measurements of the Final Sending End Impedance.—We have already shown in § 13 that by measuring the final sending end impedance $Z_1 = V_1/I_1$ both with the far end of the cable open and closed so as to obtain Z_f and Z_c we can find the vector impedance and admittance $R + jpL$ and $S + jpC$. Since

$$R+jpL=\frac{\sqrt{Z_fZ_c}}{l}\tanh^{-1}\sqrt{\frac{Z_c}{Z_f}}$$

$$S+jpC=\frac{1}{\sqrt{Z_fZ_c}}\frac{1}{l}\tanh^{-1}\sqrt{\frac{Z_c}{Z_f}}.$$

These last quantities are therefore obtained in the form of complex quantities $a+jb$ and can be drawn as vectors. Hence we see at once that the horizontal steps of the two vectors give us the values respectively of R and S and the two vertical steps the values of pL and pC, from which L and C can be obtained since $p = 2\pi n$ is known. Thus the four constants of the cable can be obtained by two measurements made with the Cohen barretter or any other means which enable us to measure the impedance of the cable when open and when short circuited or, which comes to the same thing, the sending end current and its phase difference and the impressed voltage in the two cases.

Thus, for instance, Messrs. Cohen and Shepherd (*loc. cit.*) measured the constants for a 10-mile length of the National Telephone Company's standard 20lb. dry core paper insulated cable and for a 10-mile length of an equivalent artificial cable at a frequency of 750 as follows:

—	Impedance in ohms.	
	Far end open.	Far end closed.
10-mile length of standard cable	$495\overline{\diagdown 54^\circ\ 20'}$	$657\overline{\diagdown 29^\circ\ 18'}$
10-mile artificial cable . .	$498\overline{\diagdown 51^\circ\ 28'}$	$644\overline{\diagdown 36^\circ\ 6'}$

From which it follows that for the

10-mile length of standard cable } $R = 82{\cdot}4 \quad L = 0{\cdot}00145 \quad C = 0{\cdot}0540 \quad S = 7{\cdot}12 \times 10^{-6}$

10-mile artificial cable } $R = 94{\cdot}0 \quad L = 0{\cdot}00020 \quad C = 0{\cdot}0624$

In practice it is best to check the values of R and C by direct measurements. Since, however, the constants are mostly required in the expressions $\sqrt{R^2 + p^2 L^2}$ and $\sqrt{S^2 + p^2 C^2}$ these can be obtained directly from the impedance measurements as single numbers.

CHAPTER VIII

CABLE CALCULATIONS AND COMPARISON OF THEORY WITH EXPERIMENT

1. Necessity for the Verification of Formulæ.—Since the object of all our investigations is to obtain rules for predetermining the performance of cables and improving their action as conductors, it is essential to test the theory and formulæ at which we have arrived by comparing the predictions of the theory with the actual results of measurement in as many cases as possible in order that we may obtain confidence in them as a means of foretelling the results in those cases in which we cannot check the measurements because the cable is not then made. Formulæ are of no use to the practical telegraph or telephone engineer unless they are reduced to such a form that they can be used for arithmetic calculations of the above kind by the aid of accessible tables.

It is essential therefore that the student in this subject should be shown how to employ the formulæ which have been obtained in numerical calculations, assuming that the necessary data and tables are available. In the last chapter of this book are given sundry data and references to published tables of various kinds. We shall proceed then to give a certain number of instances of calculation and verification of formulæ.

2. To Calculate the Current at any Point in a Cable Earthed or Short Circuited at the Far End when a simple Periodic Electromotive Force is applied at the Sending End.—The formula required for this purpose is proved in Chapter III., § 2, equation (25).

It is as follows:

$$I=I_1 \text{ Cosh } Px-\frac{V_1}{Z_0}\text{Sinh } Px \qquad . \quad . \quad . \quad (1)$$

where x is the distance from the sending end, I is the current at this point, I_1 the current at the sending end, P the propagation constant, such that $P = a + j\beta$, and Z_0 is the initial sending end or line impedance

$$=\frac{\sqrt{R+jpL}}{\sqrt{S+jpC}}=\frac{R+jpL}{a+j\beta}.$$

The details of the following measurements made with an artificial cable by Mr. H. Tinsley have been communicated by him to the author. These measurements were made with a Drysdale-Tinsley alternate current potentiometer as described in the previous chapter. The cable was equivalent to a submarine cable having a length of 230 nauts (nautical miles). The total conductor resistance was 1,440 ohms and the total capacity 72 microfarads. The inductance and leakance were negligible. Hence for this cable we have the constants

Resistance per naut $R = \frac{1440}{230} = 6{\cdot}26$ ohms.

Capacity per naut $C = \frac{72}{230\times 10^6} = \frac{0{\cdot}313}{10^6}$ farads.

An alternating electromotive of 1 volt of sine curve form was applied at one end of the cable, the far end being earthed. The frequency of the $E.M.F.$ was $n = 50$. Hence $p = 2\pi n = 314$. Accordingly $Cp = \frac{98}{10^6}$ per naut.

Since L and S are negligible we have for the attenuation and wave length constants the values

$$a=\beta=\sqrt{\frac{1}{2}CpR} = 0{\cdot}0175 \text{ per naut.}$$

Also the initial sending end impedance $Z_0 = \frac{\sqrt{R}}{\sqrt{jpC}}$. Hence $(Z_0) = 252{\cdot}8$ ohms.

The propagation constant $P = a + j\beta$.

Hence $P = 0{\cdot}0175 + j\,0{\cdot}0175$.

The sending end current I_1 under an $E.M.F.$ of 1 volt was $0{\cdot}003916$ ampere, and this is so nearly equal to $\frac{1}{252{\cdot}8}$ that it shows that $I_1 = \frac{V_1}{Z_0}$ nearly. In other words the cable is for all practical purposes extremely long. Hence the formula (1) for the current may be written in this case

$$\begin{aligned} I &= I_1 (\text{Cosh } Px - \text{Sinh } Px) \\ &= I_1 \, \epsilon^{-Px} = I_1 \, \epsilon^{-(a+j\beta)x} \\ &= I_1 (\text{Cosh } ax - \text{Sinh } ax)(\text{Cos } \beta x - j \text{ Sin } \beta x) \quad . \quad . \quad (2) \end{aligned}$$

Accordingly the strength of the current at any distance x is I_1 (Cosh ax — Sinh ax) amperes and the phase lags an angle βx behind the current at the sending end.

If then we insert in the above formula $a = 0{\cdot}0175$ and $I_1 = 0{\cdot}003916$ and give x various values, say 10, 20, 30, 100, 230, etc., we shall have the predetermined values of the current in magnitude and phase. This has been done in the table below.

TABLE I.

PREDETERMINATION OF THE CURRENT AT VARIOUS DISTANCES IN NAUTS IN THE TINSLEY ARTIFICIAL CABLE FOR WHICH $a = \beta = 0{\cdot}0175$.

x = distance in nauts from sending end.	ax = attenuation × distance.	Cosh ax − Sinh ax.	I = current in amps.	βx = phase angle in degrees.
10	·175	0·8395	0·0033	10°
20	·35	0·7047	0·00273	20°
30	·525	0·5910	0·00231	30°
40	·70	0·4967	0·00194	40°
50	·875	0·4268	0·00166	50°
100	1·75	0·1747	0·00068	100°
150	2·625	0·0723	0·00028	150°
230	4·025		0·00014	233°

As a check on the above formula the predictions in the above table may be compared with Mr. Tinsley's actual measurements. He measured the current strength, and phase difference between the current at any point and the sending end current, and set them off in a vector diagram shown in Fig. 1, in which the length of each line drawn from the origin represents in magnitude and direction the strength and phase of the current at the distances marked on it. On comparing these numbers with those in

Table I. it will be seen how nearly they agree. The formula therefore may be regarded as verified within the limits of errors of experiment.

It may perhaps be worth while to explain in detail how each current value is calculated. Taking say the distance of 20 nauts. We have $\alpha=\beta=0{\cdot}0175$. Hence $\alpha x=\beta x=20\times 0{\cdot}0175=0{\cdot}35$. We look out in the Tables of Hyperbolic Sines and Cosines Cosh 0·35 and Sinh 0·35 and find respectively 1·0618778 and 0·3571898. Their difference is 0·7047.

Multiplying this by 0·003916 amp. we have 0·0033 amp.,

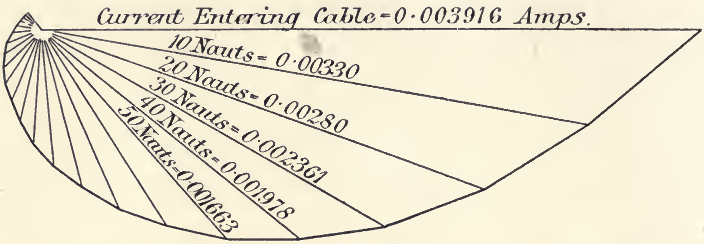

FIG. 1.—Vector Diagram of Current at various distances along an Artificial Cable.

which gives us the current in the cable at 20 nauts. The phase angle is 0·35 radians or 20°. Similarly for the other values.

3. To Calculate the Current at any Point in a Cable having a Receiving Instrument of Known Impedance at the Far End.—In this calculation the first step is to find the final sending end impedance Z_1 and final receiving end impedance Z_2 given the initial sending end impedance Z_0 and the impedance Z_r of the receiving instrument.

From equations (61) and (62) in Chapter III., § 5, we have

$$Z_1=\frac{V_1}{I_1}=Z_0\frac{Z_r\operatorname{Cosh}Pl+Z_0\operatorname{Sinh}Pl}{Z_0\operatorname{Cosh}Pl+Z_r\operatorname{Sinh}Pl}\quad . \quad . \quad . \quad (3)$$

$$Z_2=\frac{V_1}{I_2}=Z_0\operatorname{Sinh}Pl+Z_r\operatorname{Cosh}Pl\quad . \quad . \quad . \quad (4)$$

and

$$\frac{I_1}{I_2}=\operatorname{Cosh}Pl+\frac{Z_r}{Z_0}\operatorname{Sinh}Pl\quad . \quad . \quad . \quad . \quad (5)$$

whilst from equation (25) in Chapter III. we have

$$I=I_1\operatorname{Cosh}Px-\frac{V_1}{Z_0}\operatorname{Sinh}Px\quad . \quad . \quad . \quad (6)$$

Therefore

$$I = V_1 \left\{ \frac{\text{Cosh } Px}{Z_1} - \frac{\text{Sinh } Px}{Z_0} \right\} \quad . \quad . \quad . \quad (7)$$

A verification of these formulæ was made for the author by Mr. B. S. Cohen by kind permission of Mr. F. Gill in the investigation laboratory of the National Telephone Company.

The cable employed was an artificial line equivalent to a length of the National Telephone Company's standard cable having the following line constants per mile.

$R = 88{\cdot}4$ ohms per loop mile. $C = 0{\cdot}055$ microfarads per loop mile. L and S negligible. The sending end electromotive force was generated by an alternator of which the frequency n was 1000 and hence $p = 2\pi n$ was 6280. Hence since L and S are zero the attenuation constant a and wave length constant β were both equal to $\sqrt{\frac{1}{2} pCR}$ or

$$a = \beta = \sqrt{\frac{1}{2} \times 6280 \times 0{\cdot}055 \times 10^{-6} \times 88{\cdot}4} = 0{\cdot}123.$$

Therefore the propagation constant

$$P = a + j\beta = 0{\cdot}123 + j\ 0{\cdot}123.$$

The initial sending end or line impedance

$$Z_0 = \frac{\sqrt{R}}{\sqrt{jpC}} = \frac{\sqrt{88{\cdot}4}}{\sqrt{j6280 \times {\cdot}055 \times 10^{-6}}} = 505 \diagdown\!\overline{45^\circ} \text{ vector ohms.}$$

Next as regards the impedance of the receiving instrument Z_r. This was measured and found to vary with the current through it as follows :

Current through receiver in milliamperes.	Impedance Zr in vector ohms of receiving instrument.
1·0	850 $\angle 66^\circ\ 40'$
2·0	900 $\angle 67^\circ\ 25'$
4·0	975 $\angle 68^\circ\ 5'$
6·0	1030 $\angle 68^\circ\ 15'$

The line was then joined up with an induction coil and receiver at either end, representing local battery subscribers' instruments, as in the diagram in Fig. 2. Alternating current at a frequency of 1000 was then sent through the line by means of one of the induction coils from a small sine wave alternator. The current at each end of the line was measured by Cohen barretters, each barretter being shunted with a 100-ohm shunt and calibrated under these conditions. The applied *E.M.F.* (V_1) at the sending end of the line was measured with an Ayrton-Mather electrostatic voltmeter and found to be 3·02 volts (*R.M.S.* value).

A line equal to a length of 15 miles of the standard cable was then employed and the currents measured at the sending and receiving ends. The ratio of the sending end to receiving end current or I_1/I_2 was found by measurement to be 5·3. The received current I_2 was found to be 1·25 milliamperes.

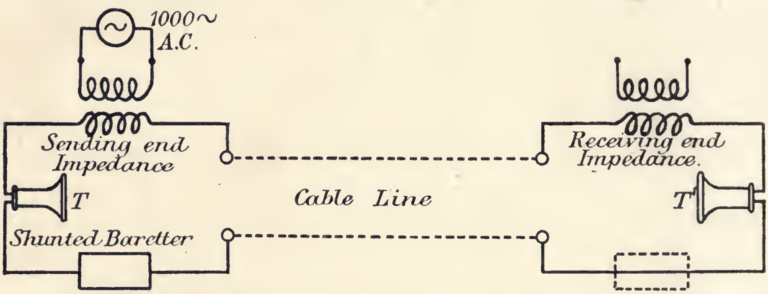

FIG. 2.—Experimental Cable with arrangements for measuring the Terminal Currents.

We may compare these numbers with the predictions of theory.

The length l of cable used was 15 miles. Therefore

$$Pl = 2{\cdot}625 \angle 45^\circ = 1{\cdot}845 + j\,1{\cdot}845.$$

Hence $\quad \text{Sinh } Pl = \text{Sinh } (1{\cdot}845 + j\,1{\cdot}845),$

and $\quad \text{Cosh } Pl = \text{Cosh } (1{\cdot}845 + j\,1{\cdot}845).$

Now 1·845 radians = 105° 44′ and the supplement of this angle is 74° 16′. We have then to calculate the value of

$$\text{Sinh } (1{\cdot}845 + j\,1{\cdot}845) = \text{Sinh } 1{\cdot}845 \text{ Cos } 105^\circ\,44' + j \text{ Cosh } 1{\cdot}845 \text{ Sin } 105^\circ\,44'$$

$$\text{Cosh } (1{\cdot}845 + j\,1{\cdot}845) = \text{Cosh } 1{\cdot}845 \text{ Cos } 105^\circ\,44' + j \text{ Sinh } 1{\cdot}845 \text{ Sin } 105^\circ\,44'$$

Now
$$\begin{aligned}\text{Sinh } 1\cdot845 &= 3\cdot0850757,\\ \text{Cosh } 1\cdot845 &= 3\cdot2431041,\\ \text{Cos } 105^\circ\,44' &= -\cdot271160,\\ \text{Sin } 105^\circ\,44' &= \cdot962534.\end{aligned}$$

Hence
$$\begin{aligned}\text{Sinh } (1\cdot845+j1\cdot845) &= -0\cdot8364+j3\cdot1215\\ &= 3\cdot231\diagdown\underline{105^\circ}.\end{aligned}$$

Also
$$\begin{aligned}\text{Cosh } (1\cdot845+j1\cdot845) &= -0\cdot8792+j2\cdot9694\\ &= 3\cdot097\diagdown\underline{106^\circ\,30'}.\end{aligned}$$

Therefore
$$\text{Tanh } (1\cdot845+j1\cdot845)=1\cdot043\overline{\diagdown 1^\circ\,30'}.$$

Also
$$Z_0=505\overline{\diagdown 45^\circ} \text{ and } Z_r=860\diagup\underline{66^\circ}\,54'.$$

Hence
$$\frac{Z_r}{Z_0}=1\cdot7\diagup\underline{111^\circ\,54'}.$$

Accordingly
$$\begin{aligned}Z_0 \text{ Sinh } Pl &= 505\overline{\diagdown 45^\circ}\times 3\cdot231\diagdown\underline{105^\circ}\\ &= 1631\diagup\underline{60^\circ}\\ &= 815+j\,1412.\end{aligned}$$

$$\begin{aligned}Z_r \text{ Cosh } Pl &= 860\diagup\underline{66^\circ\,54'}\times 3\cdot097\diagdown\underline{106^\circ\,30'}\\ &= 2663\diagdown\underline{173^\circ\,34'}\\ &= -2646+j298.\end{aligned}$$

Hence
$$\begin{aligned}Z_2=\frac{V_1}{I_2} &= Z_0 \text{ Sinh } Pl+Z_r \text{ Cosh } Pl=-1831+j1710\\ &= 2500\diagdown\underline{133^\circ\,40'}.\end{aligned}$$

Furthermore
$$\begin{aligned}Z_0 \text{ Cosh } Pl &= 505\overline{\diagdown 45^\circ}\times 3\cdot097\diagdown\underline{106^\circ\,30'}\\ &= 1564\diagup\underline{61^\circ\,30'}\\ &= 746+j1374.\end{aligned}$$

$$\begin{aligned}Z_r \text{ Sinh } Pl &= 860\diagup\underline{66^\circ\,54'}\times 3\cdot231\diagup\underline{105^\circ}\\ &= 2778\diagup\underline{171^\circ\,54'}\\ &= -2750+j390.\end{aligned}$$

Hence
$$\begin{aligned}Z_0 \text{ Cosh } Pl+Z_r \text{ Sinh } Pl &= -2004+j1764\\ &= 2667\diagup\underline{138^\circ\,35'}.\end{aligned}$$

Now
$$\begin{aligned}Z_1 &= Z_0\,\frac{Z_r \text{ Cosh } Pl+Z_0 \text{ Sinh } Pl}{Z_0 \text{ Cosh } Pl+Z_r \text{ Sinh } Pl}\\ &= 505\overline{\diagdown 45^\circ}\,\frac{2500\diagdown\underline{133^\circ\,40'}}{2667\diagdown\underline{138^\circ\,35'}}\\ &= 473\overline{\diagdown 49^\circ\,55'}.\end{aligned}$$

Accordingly the four impedances are

$Z_0 = 505 \diagdown\overline{45^\circ}$ = line impedance or initial sending end impedance,

$Z_1 = 473 \diagdown\overline{49^\circ\ 55'}$ = final sending end impedance.

$Z_2 = 2500 \diagdown\underline{133^\circ\ 40'}$ = final receiving end impedance,

$Z_r = 860 \underline{\diagup 66^\circ\ 54'}$ = receiving instrument impedance.

Now the impressed or sending end voltage was 3·02 volts. Therefore we have

$$I_1 = \frac{V_1}{Z_1} = \text{sending end current} = \frac{3{\cdot}02}{473} = 0{\cdot}0064 \text{ amp.},$$

$$I_2 = \frac{V_1}{Z_2} = \text{received current} = \frac{3{\cdot}02}{2500} = 0{\cdot}001208 \text{ amp.},$$

$$\frac{I_1}{I_2} = \frac{6400}{1208} = 5{\cdot}3 \text{ by calculation.}$$

The ratio $\frac{I_1}{I_2}$ was also found to be 5·3 by observation.

The received current $I_2 = 1{\cdot}208$ milliamperes by calculation, and was found to be 1·25 milliamperes by observation.

Hence there is a very good agreement between the observed values and those predicted by our formulæ, which are thereby confirmed.

An additional illustration of the above formulæ may be given as follows:

Suppose a length of ten miles of the same standard cable to have a plain Bell receiving telephone placed across the receiving end, we can then calculate the current through this receiver as follows:

The received current $I_2 = \dfrac{V_1}{Z_0 \text{ Sinh } Pl + Z_r \text{ Cosh } Pl.}$

In this case we have for a ten-mile length of the cable

$$Z_0 = 465 - j415 = 625 \diagdown\overline{41^\circ\ 45'} \text{ ohms,}$$

and Z_r = for a 60-ohm Bell receiver is given approximately by the formula

$$Z_r = 134 + j91 = 162 \underline{\diagup 34^\circ\ 15'} \text{ ohms.}$$

We can then easily find that

$$Pl = 10 \times 0{\cdot}1 + j10 \times 0{\cdot}1 = 1 + j1,$$

and hence $\quad \text{Sinh } Pl = 0{\cdot}634 + j1{\cdot}297 = 1{\cdot}445 \underline{\diagup 64^\circ},$

and $\quad \text{Cosh } Pl = 0{\cdot}83 + j{\cdot}99 = 1{\cdot}292 \underline{\diagup 50^\circ\ 15'}.$

Hence $\quad Z_r \text{ Cosh } Pl = 20 + j207 = 209 \underline{/84° \, 30'}$

and $\quad Z_0 \text{ Sinh } Pl = 833 + j341 = 900 \underline{/22° \, 15'}$.

Therefore

$$Z_r \text{ Cosh } Pl + Z_0 \text{ Sinh } Pl = 853 + j548 = 1014 \underline{/32° \, 40'}.$$

Accordingly $\quad I_2 = \dfrac{V_1}{1014} = \dfrac{10}{1014} = 9{\cdot}8$ milliamperes.

The reader should notice that as Pl increases, that is as the length of the cable increases, the values of Sinh Pl and Cosh Pl approximate. Since Sinh 4 is nearly equal to Cosh 4, and since a and β for the standard cable are equal to about 0·1, it follows that for cable lengths of forty miles and upwards we can greatly simplify the formulæ by writing Sinh Pl = Cosh Pl in them. Thus under these conditions we have the receiving end impedance Z_2 given by

$$Z_2 = (Z_0 + Z_r) \text{ Sinh } Pl \quad . \quad . \quad . \quad . \quad (8)$$

and the received current I_2 by the reduced formula

$$I_2 = \frac{V_1}{(Z_0 + Z_r) \text{ Sinh } Pl} \quad . \quad . \quad . \quad . \quad (9)$$

and the sending end current by

$$I_1 = \frac{V_1}{Z_0} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the ratio I_1/I_2 by

$$\frac{I_1}{I_2} = \frac{Z_0 + Z_r}{Z_0} \text{ Sinh } Pl \quad . \quad . \quad . \quad . \quad (11)$$

Thus for forty miles of standard cable we have $l = 40$, $al = 4 = \beta l$, and Sinh Pl = Sinh 4 (Cos 4 + j Sin 4) = 27·3 (Cos 4 + j Sin 4).

Now for the same cable and receiving instrument we have

$$\begin{aligned} Z_0 &= 465 - j415 \\ Z_r &= 134 + j\,91 \\ \hline Z_0 + Z_r &= 599 - j324 = 681 \overline{\diagdown 28° \, 35'} \end{aligned}$$

and $\quad \text{Sinh } Pl = 27{\cdot}3 \underline{/229° \, 20'}$.

Hence $\quad Z_2 = (Z_0 + Z_r) \text{ Sinh } Pl = 18591{\cdot}3 \underline{/200° \, 45'}$

and $\quad Z_1 = Z_0 = 623 \overline{\diagdown 41° \, 45'}$.

Hence for $V_1 = 10$ $\quad I_1 = \dfrac{1}{62}$ ampere and $I_2 = \dfrac{1}{1860}$ ampere.

As regards the ratio of I_1/I_2 or of the sending end to receiving end currents, we have always

$$\frac{I_1}{I_2} = \text{Cosh } Pl + \frac{Z_r}{Z_0} \text{Sinh } Pl \quad . \quad . \quad . \quad (12)$$

If Pl is very small, approximating to zero, because the length l

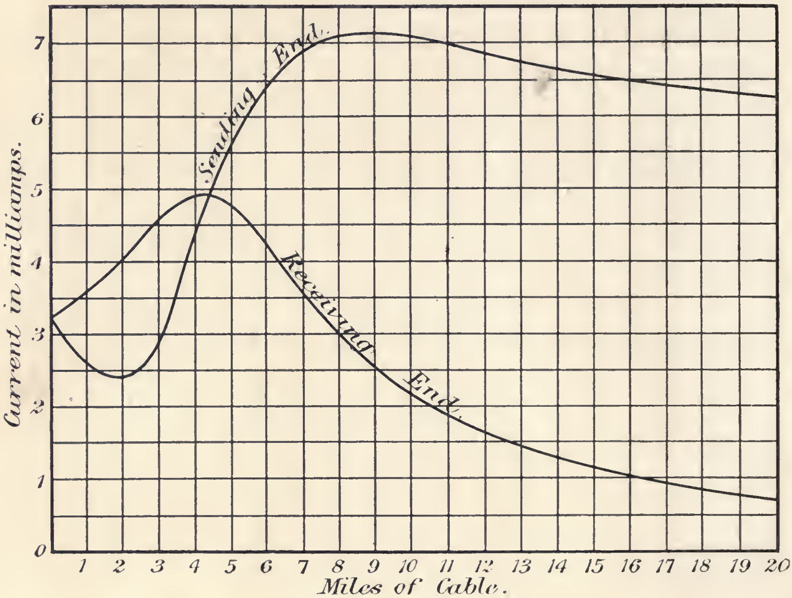

Fig. 3.—Curves showing the variation of the sending end and receiving end Currents in a Telephonic Cable (Cohen).

is small, then Cosh $Pl = 1$ Sinh $Pl = 0$ and $I_1/I_2 = 1$, as it should be.

If Pl is very large, say, greater than 4, because l is large, then Cosh Pl = Sinh Pl, and we have

$$\frac{I_1}{I_2} = \left(1 + \frac{Z_r}{Z_0}\right) \text{Cosh } Pl \quad . \quad . \quad . \quad . \quad (13)$$

By equation (74) of § 5, Chapter III., this equation for the ratio I_1/I_2 generally may be written

$$\frac{I_1}{I_2} = \frac{\text{Cosh } (Pl + \gamma)}{\text{Cosh } \gamma} \quad . \quad . \quad . \quad . \quad (14)$$

where $\gamma = \tanh^{-1} \dfrac{Z_r}{Z_0}$.

For certain values of γ and Pl it is possible for Cosh $(Pl+\gamma)$ considered as a vector to have a smaller size than that of Cosh γ. If γ and P are kept constant and l varied, then for some values of γ and P we shall have the ratio I_1/I_2 equal to, less than, and greater than unity as l progressively increases.

This signifies that the current at the receiving end may, under certain conditions, be greater than the current at the sending end. This takes place when l is small, and increasing from zero.

This variation in the ratio of I_1/I_2, or of the sending end to the receiving end current, as the length of the cable increases, is well shown by the observations, represented by the curves in Fig. 3, which were taken by Mr. B. S. Cohen in the Investigation Laboratory of the National Telephone Company. For various lengths of standard telephone cable and for the same receiving instrument the currents I_1 and I_2 were measured with two barretters, and the observed values are represented by the firm line curves for various lengths of cable. It will be seen that when the length of cable is zero the two currents are identical, as they should be. As the length of cable increases up to about four miles the current at the receiving end is greater than that at the sending end. At a length of about 4·4 miles the two currents are again equal. Beyond that length the sending end current is greater than the receiving end current.

4. To calculate the Voltage at the Receiving End of a Cable when open or insulated, and the Current when closed or short circuited.—The formulæ in this case are

$$V_2 = V_1 \text{ Sech } Pl \qquad . \quad . \quad . \quad . \quad (15)$$

$$I_2 = \frac{V_1}{Z_0} \text{ Cosech } Pl \qquad . \quad . \quad . \quad . \quad (16)$$

where V_1 is the impressed voltage at the sending end, and V_2 and I_2 the voltage and current at the receiving end.

Thus suppose that $V_1 = 10$ volts, and that we have to deal with twenty miles of standard cable for which $\alpha = \beta = 0{\cdot}1$ nearly. Then $Pl = 20\,\alpha + j20\,\beta = 2 + j2$. Then from the table we have

$$\text{Cosh } 2 = 3{\cdot}76, \qquad \text{Sinh } 2 = 3{\cdot}627,$$
$$\text{Cos } 2 = -0{\cdot}416, \qquad \text{Sin } 2 = 0{\cdot}909,$$

since an angle of two radians $= 114^\circ\ 35'\ 30''$.

Hence $\quad \text{Cosh } Pl = -3{\cdot}76 \times {\cdot}416 + j3{\cdot}627 \times {\cdot}909$

$$= -1{\cdot}564 + j3{\cdot}297$$

$$= 3{\cdot}65 \underline{/115^\circ\ 18'},$$

and $\quad \text{Sinh } Pl = -3{\cdot}627 \times {\cdot}416 + j3{\cdot}76 \times {\cdot}909$

$$= -1{\cdot}51 + j3{\cdot}42$$

$$= 3{\cdot}74 \underline{/114^\circ\ 12'}.$$

Therefore $\quad \text{Sech } Pl = 0{\cdot}273 \overline{\backslash 115^\circ\ 18'},$

$$\text{Cosech } Pl = 0{\cdot}266 \overline{\backslash 114^\circ\ 12'}.$$

Hence $\quad V_2 = 10 \times 0{\cdot}273 = 2{\cdot}73$ volts.

Then $\quad Z_0 = \dfrac{\sqrt{R + jpL}}{\sqrt{S + jpC}}$ and $\dfrac{1}{Z_0} = \dfrac{\alpha + j\beta}{R + jpL}$,

or $\quad \dfrac{1}{Z_0} = \dfrac{\alpha R + \beta pL}{R^2 + p^2L^2} + j\, \dfrac{\beta R - \alpha pL}{R^2 + p^2L^2}$

$$= \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{R^2 + p^2L^2}} \underline{\Big/ \tan^{-1} \frac{\beta R - \alpha pL}{\alpha R + \beta pL}}.$$

For the standard cable $R = 88$ ohms, and $L = {\cdot}001$ henry, and if we take $p = 5{,}000$ we have $pL = 5$ and $\sqrt{R^2 + p^2 L^2} = 88{\cdot}1$. Also $\sqrt{\alpha^2 + \beta^2} = 0{\cdot}1414$, and therefore $\dfrac{1}{Z_0} = 0{\cdot}0016 \underline{/41^\circ\ 45'}$. Therefore we have

$$I_1 = 10 \times {\cdot}0016 \times {\cdot}266 = 0{\cdot}004256.$$

Hence for an impressed voltage of 10 volts the voltage at the far end is 2·73 volts if the receiving end is open, and the current is 4·25 milliamperes if the receiving end is short-circuited.

5. Calculation and Predetermination of Attenuation Constants.—The predetermination of the attenuation constant α of a given type of telephone cable is a most important matter, because it is the value of this quantity that determines the speaking qualities of the cable. The fundamental formula for α is,

$$\alpha = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2 + p^2L^2)(S^2 + p^2C^2)} + RS - p^2LC\right\}} \quad . \quad (17)$$

In this formula R must be given in ohms, L in henrys, C in farads, and S in mhos or the reciprocal of ohms, and p is 2π times the frequency of the current.

Mr. H. R. Kempe has pointed out[1] that this formula is not very convenient for calculation, because in the majority of cases the quantity $\sqrt{(R^2 + p^2L^2)(S^2 + p^2C^2)} + RS$ is so nearly equal to p^2LC that a large error may be made in taking their difference unless each is worked out to many decimal places. Also it is more convenient to have a formula in which we can insert the value of R in ohms, C, in microfarads, L in millihenrys, and the reciprocal of S in ohms; that is the insulation resistance per mile, naut, or kilometre in ohms, as given directly by measurements. He has therefore changed the above expression for a into another equivalent one as follows:—

$$a=\frac{\sqrt{C}}{20}\sqrt{\sqrt{R^2+(5L)^2}-5L+\frac{200R}{rC}+0{\cdot}000128\sqrt{R^2+(5L^2)}} \quad . \quad (18)$$

In the above formula p is taken as 5000 and C is to be understood as the capacity in microfarads, L as the inductance in millihenrys, R as the copper resistance in ohms, and r as the insulation resistance in ohms, all per mile or per kilometre as the case may be.

If the cable is a loaded cable then the value of R is the conductor resistance per mile *plus* the effective resistance of the loading coils per mile and the value of L is the inductance per mile of the cable *plus* that of the loading coils per mile reckoned in millihenrys.

In the case of well-constructed loading coils the effective resistance is about 6 ohms for every 100 millihenrys of inductance. In the case of the cable itself the inductance will be about 1 millihenry per mile. For some types of dry core land cable the value of the insulation conductivity S is so small that it can be neglected. Under these conditions we have

$$a=\sqrt{\frac{Cp}{2}\left\{\sqrt{R^2+p^2L^2}-pL\right\}} \quad . \quad . \quad . \quad (19)$$

For unloaded cables, and for a frequency such that $p = 5000$, we shall generally have R greater than pL, or at least not very different from it.

[1] See Appendix X to a paper by Major W. A. J. O'Meara, C.M.G., on "Submarine Cables for Long Distance Telephone Circuits," *Journal Inst. Elec. Eng. Lond.*, Vol. XLVI., p. 309, 1911.

There is then no difficulty in finding the value of

$$\sqrt{R^2+p^2L^2}-pL$$

with a fair amount of accuracy. If, however, L is large as in the case of loaded artificial cables, then, as we have already shown in Chapter IV.,

$$\sqrt{R^2+p^2L^2}-pL=\frac{R^2}{2pL}.$$

Hence when pL/R is large and $S = 0$ we have the value of the attenuation constant a given by the expression (see p. 297),

$$a=\frac{R}{2}\sqrt{\frac{C}{L}} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

When S is not absolutely zero then a somewhat more accurate approximation is given by the expression

$$a=\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{S}{2}\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad (21)$$

$$=\frac{1}{2}\left(R+L\frac{S}{C}\right)\sqrt{\frac{C}{L}} \quad . \quad . \quad . \quad . \quad (22)$$

If the leakance S can be neglected, but if the inductance L is small, even as small as one millihenry per mile, it is preferable to calculate the attenuation constant by the formula

$$a=\sqrt{\frac{1}{2}\,Cp\,(\sqrt{R^2+p^2L^2}-Lp)} \quad . \quad . \quad . \quad (23)$$

rather than by the formula

$$a=\sqrt{\frac{1}{2}\,CpR} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

As an example of the difference the following values may be given, which were furnished by Mr. A. W. Martin of the General Post Office in a discussion at the Physical Society on a paper by Professor J. Perry on "Telephone Circuits."[1] The figures show that for the constants given the inductance of the cable though small should be taken into account in the calculation.

The value of L, the inductance per mile of various types of cable, is approximately as follows :—

$L = 0{\cdot}001$ henry per mile for underground cables.

$L = 0{\cdot}0017$,, ,, ,, submarine cables.

$L = 0{\cdot}0032$ to $0{\cdot}0042$ for aerial copper wire lines.

[1] See *The Electrician*, Vol. LXIV., p. 880, March 11, 1910, for Mr. Martin's remarks, and *Proceedings of the Physical Society*, Vol. XXII., p. 252, 1910, for Prof. Perry's paper on "Telephone Circuits."

TABLE II.

TABLE OF ATTENUATION CONSTANTS (a) CALCULATED AND OBSERVED. $p=2\pi n=5000$.

Constants of the Cable per mile.			Attenuation Constant (α) calculated by Equation (24).	Attenuation Constant (α) calculated by Equation (23).	Attenuation Constant (α) observed.
R ohms.	C mfds.	L henrys.			
88	0·050	0·001	0·105	0·102	—
88	0·054	0·001	0·109	0·106	0·106
18	0·055	0·001	0·050	0·043	0·046
12	0·065	0·001	0·044	0·036	0·037

In practice it is found that the value of S/C is very far from being negligible when inductance is introduced into the cable. Hence leakance acts to increase attenuation. It is thus easily seen that in the case of loaded cables any large amount of dielectric conductivity or small insulation resistance has a great effect in increasing the attenuation constant. Certain dielectrics such as gutta percha are well known to have a low dielectric resistance and hence create a relatively large attenuation constant in cables insulated with them.

It has been stated that this large value of S in the case of gutta percha insulated wire would nullify the effect of any loading by inductance.[1] This, however, was disproved by experiments made by Major O'Meara, Engineer-in-Chief to the General Post Office, and described by him in a paper on Submarine Cables for Long Distance Telephone Circuits in the following words[2]:—

"In order to settle the point definitely, it was decided to carry "out some experiments. The Department had a large stock of "No. 7 gutta percha covered wire (weight of copper, 40 lbs. per "mile; of gutta percha, 50 lbs. per mile; resistance, 44 ohms "per loop mile; electrostatic capacity wire to wire, 0·13 micro-"farad per mile), and also a number of inductance coils "(inductance, 83 millihenrys; resistance, 13·4 ohms at 750

[1] See *Elektrotechnische Zeitschrift*, Vol. XXIX., 1908, p. 588.

[2] See *Journal Institution Electrical Engineers, London*, Vol. XLVI., 1911, p. 309.

"periods per second), which had been used originally for carry-
"ing out some experiments in connection with the improvement
"of transmission of speech in subterranean cables between
"Liverpool and Manchester. Calculations were made to
"ascertain the best disposition of the coils in this particular
"type of cable—although neither the coils nor the cable were
"really of the most suitable type—and it was found that in
"order to provide 55 millihenrys per mile they should be
"inserted at intervals of $1\frac{1}{2}$ miles. A large number of speech
"tests were made on loaded circuits formed by means of the
"No. 7 gutta percha wire, by myself, Messrs. H. Hartnell,
"A. W. Martin, and other members of my staff. It was
"gratifying to find that the actual improvement in transmission
"was in complete agreement with the estimates based on the
"calculations that had been made. (By calculation the attenua-
"tion was 0·0427 per mile, and the observed result was 0·0419
"per mile.) We found that commercial speech was certainly
"practicable on 105 miles of this particular type of 'coil'
"loaded gutta percha wire, and our doubts as to the feasibility
"of the 'non-uniform' loading for submarine cables of moderate
"length were set at rest."

In the case of loaded cables the calculation of the attenuation constant can be carried out by the aid of Campbell's formula given in § 8 equation 63 of Chapter IV. This formula is, however, very troublesome to work with owing to the necessity of calculating an inverse hyperbolic function that is the value of Cosh^{-1} or Sinh^{-1} for some vector.

If the loading coils are placed at such intervals that there are nine or ten per wave reckoned by assuming that the total resistance and total inductance per mile, including that of the cable itself and of the loading coils, are distributed uniformly, and also assuming a frequency such that $p = 5000$, then if the value of $2\pi/\beta$ where β is the wave length constant is at least nine times the interval between the loading coils, we may assume that the attenuation constant a will be given sufficiently for all practical purposes by a calculation made in the usual manner with this uniformly distributed resistance and inductance. An illustration will make this clear :

A paper insulated cable had a resistance per kilometre of 27·96 ohms, a capacity per kilometre of 0·07455 microfarad, and an inductance per kilometre of 0·00056 henry. Loading coils each of 15 ohms (effective) resistance and a total or double inductance of 0·225 henry were inserted at intervals of 1·2 kilometres. It is required to find the true attenuation constant for a frequency n such that $2\pi n = p = 5000$.

We have $R = 27{\cdot}96$, $C = 0{\cdot}07455 \times 10^{-6}$, $L = 0{\cdot}00056$; $S = 0$ and $p = 5000$.

For the line proper the propagation constant P where $P = a + j\beta$, and a and β are calculated from the usual formulæ,

$$a=\sqrt{\frac{Cp}{2}\left\{\sqrt{R^2+p^2L^2}-Lp\right\}},\ \beta=\sqrt{\frac{Cp}{2}\left\{\sqrt{R^2+p^2L^2}+Lp\right\}}$$

is obtained by inserting in the above expressions the values of the R, L and C for the line itself. Hence we obtain

$$P=0{\cdot}06867+j\,0{\cdot}07589=0{\cdot}10234\,\underline{/47^\circ\ 51{\cdot}5'}.$$

Now the coil interval $d = 1{\cdot}2$ kilometres. Hence

$$Pd=0{\cdot}12281\,\underline{/47^\circ\ 51{\cdot}5'}=0{\cdot}082402+j\,0{\cdot}091062.$$

Again for the line

$$Z_0=\frac{\sqrt{R+jpL}}{\sqrt{S+jpC}}=274{\cdot}74\,\overline{\diagdown 42^\circ\ 8{\cdot}4'}.$$

Now

$$\begin{aligned}\text{Cosh } Pd&=\text{Cosh }(0{\cdot}082402+j\,0{\cdot}091062)\\&=\text{Cosh } 0{\cdot}082402 \text{ Cos } 0{\cdot}091062+j \text{ Sinh } 0{\cdot}082402 \text{ Sin } 0{\cdot}091062\\&=0{\cdot}999173+j\,0{\cdot}007499.\end{aligned}$$

Also

$$\text{Sinh } Pd=0{\cdot}082146+j\,0{\cdot}091219=0{\cdot}122347\,\underline{/47^\circ\ 59{\cdot}8'}.$$

The loading coil impedance $= Z' = R' + jpL'$ is equal to

$$15+j\,1125=1125{\cdot}1\,\underline{/89^\circ\ 14'}.$$

Also $$2Z_0=549{\cdot}48\,\overline{\diagdown 42^\circ\ 8{\cdot}4'}.$$

Hence $$\frac{Z'}{2Z_0}=2{\cdot}0476\,\underline{\diagdown 131^\circ\ 22{\cdot}4'}$$

and $$\frac{Z'}{2Z_0}\text{Sinh } Pd=0{\cdot}25052\,\underline{\diagdown 179^\circ\ 22{\cdot}2'}$$

$$= -0{\cdot}25050+j\,0{\cdot}0027532.$$

By Campbell's formula (see Chapter IV.) if P' is the effective Propagation constant of the loaded line we have

$$\text{Cosh } P'd = \text{Cosh } Pd + \frac{Z'}{2Z_0} \text{ Sinh } Pd.$$

Therefore $\quad \text{Cosh } P'd = 0{\cdot}74867 + j\, 0{\cdot}010252.$

Therefore $\quad P'd = \text{Cosh}^{-1} \{0{\cdot}74867 + j0{\cdot}010252\}$

By the formula in § 5, Chapter I., we have then

$$P'd = \text{Cosh}^{-1} (1{\cdot}000120) + j \text{ Cos}^{-1} (0{\cdot}74858)$$
$$= 0{\cdot}0155 + j\, 0{\cdot}7249.$$

But $\quad d = 1{\cdot}2$ kilometres. Hence

$$P' = 0{\cdot}0129 + j\, 0{\cdot}604$$
$$= \alpha' + j\beta'$$

where α' is the effective attenuation constant of the loaded line.

Accordingly $\quad \alpha' = 0{\cdot}0129$

and $\quad \beta' = 0{\cdot}604$

Therefore the wave length $\lambda' = \dfrac{2\pi}{\beta'}$ and $\lambda' = 10{\cdot}4$ kilometres. There are therefore $10{\cdot}4/1{\cdot}2 = 9$ loading coils per wave, and the spacing is by Pupin's law sufficiently close.

Suppose then that the total resistance and total inductance of all the coils is smoothed out and added to that of the line, we shall have a total resistance of 27·96 ohms per kilometre of line and 15 ohms due to the loading coil per 1·2 kilometre or $15/1{\cdot}2 = 12{\cdot}4$ ohms per kilometre. Hence a total resistance (R'') per kilometre of $27{\cdot}96 + 12{\cdot}4 = 40{\cdot}36$ ohms.

In the same way the total smoothed out inductance L'' per kilometre is $0{\cdot}00056 + 0{\cdot}225/1{\cdot}2 = 0{\cdot}18806$ henry. If then we calculate the attenuation constant α'' and wave length constant β'' for this smoothed out cable having a total resistance $R'' = 40{\cdot}36$ ohms per kilometre and a total inductance $L'' = 0{\cdot}18806$ henrys per kilometre and capacity $C = 0{\cdot}07455 \times 106^{-6}$ farads per kilometre, using the formulæ

$$\alpha'' = \sqrt{\frac{Cp}{2}\left\{\sqrt{R''^2 + p^2L''^2} - L''p\right\}} \quad . \quad . \quad . \quad (25)$$

$$\beta'' = \sqrt{\frac{Cp}{2}\left\{\sqrt{R''^2 + p^2L''^2} + L''p\right\}} \quad . \quad . \quad . \quad (26)$$

we find we obtain values

$$\alpha'' = 0{\cdot}0128 \qquad \beta'' = 0{\cdot}590.$$

The smoothed out attenuation constant α'' is therefore very nearly equal to the effective attenuation constant α' as calculated by Campbell's formula. It has been shown by Mr. G. A. Campbell that if the spacing of the coils is such that there are fewer than 9 coils per wave, then the actual attenuation constant α' of the loaded line is greater than that predicted by assuming the total resistance and inductance smoothed out (α'') in the following proportions[1] :—

For 8 coils per wave	α' is	greater	than α''	by		1%
„ 7	„	„	„	„		2%
„ 6	„	„	„	„		3%
„ 5	„	„	„	„		7%
„ 4	„	„	„	„		16%
„ 3	„	„	„	„		200% or more.

As a rule, therefore, in calculating the attenuation of loaded lines we can proceed as follows. Assume the total resistance and inductance of the line and the loading coils to be smoothed out and uniformly distributed and calculate the resulting R, L, and C per mile or per kilometre of line. Then find the wave length constant β and the wave length $\lambda = 2\pi/\beta$ for the highest frequency to be used in practice or for the average frequency (800) of the speaking voice. If this wave length λ is more than eight or nine times the distance between the loading coils, then we may proceed to calculate the attenuation constant with this smoothed out resistance and inductance, and the resulting value will be quite near enough to the actual measured or real attenuation constant. We thus avoid the troublesome calculations involved in using the Campbell formula.

As an example of this calculation we may take the loaded Anglo-French telephone cable laid in 1910 by the General Post Office, which is furthermore described in the next chapter of this book. The constants of this cable as given by Major O'Meara are as follows:

CONSTANTS OF THE UNLOADED CABLE.

$R=14{\cdot}42$	ohms	per knot or nautical mile of loop.			
$L=0{\cdot}002$	henrys	„	„	„	„
$C=0{\cdot}138$	microfarad	„	„	„	„
$K=2{\cdot}4\times10^5$	mhos	„	„	„	„
$n=750$ $p=2\pi n=4710.$					

[1] See Dr. A. E. Kennelly, "The Distribution of Pressure and Current over Alternating Current Circuits," *Harvard Engineering Journal*, 1905—1906.

The cable was loaded with coils having an effective resistance of 6 ohms at 750 *p.p.s.* and an inductance of 100 millihenrys. These coils were placed 1 knot (naut. mile) apart. Hence the constants of the loaded cable were

$R=20{\cdot}45$	ohms	per knot loop of cable.
$L=0{\cdot}1$	henry	,, ,, ,,
$C=0{\cdot}138$	microfarad	,, ,, ,,
$S=2{\cdot}4\times 10^{-5}$	mhos	,, ,, ,,

Hence for $n = 750$ and $p = 4710$ we have

$$\sqrt{R^2+p^2L^2} = \sqrt{418+221841}. \quad \text{Also}$$

$$\sqrt{S^2+p^2C^2} = 10^{-6}\sqrt{576+422500}.$$

Again we have $\sqrt{LC}=\sqrt{\dfrac{138}{10^{10}}}, \quad Lp=471, \quad Cp=\dfrac{65}{10^5}$

Accordingly the wave length constant

$$\beta=p\sqrt{LC}=4710\sqrt{\frac{138}{10^{10}}}=0{\cdot}542,$$

and the wave length $\lambda = 2\pi/\beta = 11{\cdot}6$ knots.

Therefore the coils are placed about 11 or 12 to the wave and fulfil the necessary condition.

Then, since R may be neglected in comparison with Lp and S in comparison with Cp, we have

$$a=\sqrt{\frac{RS}{2}}=\sqrt{\frac{245}{10^6}}=0{\cdot}0166.$$

The measured value was found to be $0{\cdot}0166$.

6. Tables and Data for assisting Cable Calculations.—The calculations necessary in connection with the subject here explained are facilitated by the possession of good mathematical tables of various kinds. The reader will have seen that part of the trouble connected with them depends upon the necessity for constantly converting the complex expression for a vector from one form, $a + jb$, into another form, $\sqrt{a^2+b^2} \,/\, \tan^{-1} b/a$, and the reverse. To add or subtract two complexes they must be thrown into the form $a + jb$, $c + jd$, and their sum and difference are then $(a + c) + j\,(b + d)$ and

$(a - c) + j(b - d)$. On the other hand, to multiply, divide, or power them they must be put into the form $A \underline{/ \theta}$, $B \underline{/ \phi}$, where $A = \sqrt{a^2 + b^2}$ and $\tan \theta = b/a$, and $B = \sqrt{c^2 + d^2}$ $\tan \phi = d/c$; and then their product or quotient is $AB \underline{/ \theta + \phi}$, $\frac{A}{B} \underline{/ \theta - \phi}$, and square root $\sqrt{A} \underline{/ \theta/2}$, etc. This process is somewhat assisted by possession of good tables of squares and square roots of numbers, or by the use of a good slide rule or of tables of four-figure logarithms.

We can then find from a and b pretty quickly $\sqrt{a^2 + b^2}$. It may also be done graphically, but with less accuracy, by drawing a right-angled triangle whose sides are a and b, and the hypothenuse is then $\sqrt{a^2 + b^2}$.

Very useful tables of squares and square roots, as well as of circular and hyperbolic functions, have been drawn up by Mr. F. Castle, and are published by Macmillan & Co., St. Martin's Street, London, W.C., entitled "Five-Figure Logarithmic and other Tables." What is really required is an extensive table of the logarithms to the base 10 of hyperbolic functions, viz., $\log_{10}$ Sinh u, $\log_{10}$ Cosh u, Log_{10} Tanh u from $u = 0$ to $u = 12$, and similar tables of $\log_{10}$ Sin θ, Log_{10} Cos θ, for various values of θ in *radians* from $\theta = 0$ to $\theta = 12$.

We then require tables of natural sines, cosines, and tangents. If the vector is given in the form $a + jb$, to convert to $A \underline{/ \theta}$ we have to find the angle θ whose tangent is b/a, and if given in the form $A \underline{/ \theta}$ we have to find $A \cos \theta + jA \sin \theta$ to convert it to the other form.

Lastly, we have to provide tables of hyperbolic functions *Sinh*, *Cosh*, *Tanh*, *Sech*, *Cosech*, and *Coth*. A table of these functions is given in the Appendix.

The most troublesome matter is the calculation of the hyperbolic function of complex angles, that is, finding the value of Cosh $(a + jb)$, Sinh $(a + jb)$, etc. No tables of these of any great range have yet been published. The author understands that such tables are in course of preparation by Dr. A. E. Kennelly, and will be extremely valuable. We require to be able to find these hyperbolic functions for any vector, so that we can enter the table with values of a and b and find at once Sinh $(a+jb)$, Cosh $(a+jb)$, etc.

At present the worker has to calculate each case for himself by the formula given in Chapter I., viz.,

Sinh $(a+jb) =$ Sinh a Cos $b + j$ Cosh a Sin b, etc., etc.

This is a tedious business, but at present there is little available assistance.

The labour can be somewhat relieved by the use of a mechanical calculator for multiplying and dividing numbers. This performs the brain-wasting labour, and the operator has then only to put the decimal point rightly.

To some small extent the calculations are relieved by the use of the tables of Sinh $(a+jb)$, etc., given in Chapter I.

The following data for various types of line and receiving instruments will be found very useful in practical calculations and proposed undertakings. They have mostly been obtained by experience and measurements made in the Investigation Laboratory of the National Telephone Company, and for permission to make use of them here the author is indebted to the courtesy of Mr. F. Gill, the Engineer-in-Chief of the National Telephone Company.

In all the following tables the standard frequency n adopted is 796 so that $2\pi n = 5{,}000$. This is sufficiently near to the average telephonic frequency to give results useful in practice.

It was agreed at the Second International Conference of Engineers of Telephone and Telegraph Administrations, held in Paris, September, 1910, that this angular velocity, $p = 5{,}000$, should be the standard one for telephonic measurements, and that these should be made with a pure sine wave curve of electromotive force.

In the following tables the abbreviations used are:—

L.B. for *local battery*. An *L.B.* instrument is one supplied with current from cells fitted locally.

C.B. means *central battery*. By a *C.B.* termination is understood the combination of a central battery telephone instrument together with exchange cord circuit apparatus which constitutes the termination of the junction or trunk line.

The following symbols are used in the tables :—

R = resistance of line per mile or per kilometre in ohms,

L = inductance of line per mile or per kilometre in henrys,

C = capacity of line per mile or per kilometre in farads,

S = dielectric conductivity per mile or per kilometre in mhos or reciprocal ohms,

P = propagation constant $= \alpha + j\beta = \sqrt{R + jpL}\,\sqrt{S + jpC}$,

α = attenuation constant,

β = wave length constant,

λ = wave length $= 2\pi/\beta$,

W = wave velocity $= p/\beta$,

Z_0 = line impedance or initial sending end impedance = $\sqrt{R + jpL}\,/\,\sqrt{S + jpC}$,

Z_r = impedance of terminal instrument,

T_r = transmission equivalent = ratio of attenuation constant of the standard line to attenuation constant of the line compared. It gives the length of the line telephonically equivalent to one mile of the standard cable.

The quantities P, Z_0, Z_r, Z_r/Z_0, are vector quantities. Hence they are expressed by stating their magnitude or size and phase angle.

The following are useful figures for terminal impedances Z_r of National Telephone Company's instruments :

L.B., *H.M.T.* instrument (*S.L.* 13), 1060 $\underline{/60^\circ}$ ohms.

No. 1 *C.B.* termination, consisting of No. 25 repeater, supervisory relay, local line, and subscriber's instrument with zero local line, 418 $\underline{/44^\circ}$ ohms.

Ditto with 300-ohm line, 730 $\underline{/30^\circ}$ ohms.

The following tables contain useful data and constants for various lines and cables :

Table I.—Data of the more Important

British

Type.	Conductor Diameter.	Primary Constants.				Propagation Constant
		R	C	L	S	P.
	mm.	ohms.	farads.	henrys.	mhos.	
Open Wires:						
40 lbs. per mile bronze .	1·27	90	$\cdot 00750 \times 10^{-6}$	$4\cdot 20 \times 10^{-3}$	10^{-6}	·0590 ∠50° 48′
70 " " " .	1·68	52	$\cdot 00786 \times 10^{-6}$	$4\cdot 00 \times 10^{-3}$	"	·0468 ∠54° 48′
100 " " copper .	2·01	18	$\cdot 00810 \times 10^{-6}$	$3\cdot 90 \times 10^{-3}$	"	·0328 ∠67° 54′
150 " " " .	2·46	11·9	$\cdot 00840 \times 10^{-6}$	$3\cdot 76 \times 10^{-3}$	"	·0306 ∠73° 10′
200 " " " .	2·85	9·0	$\cdot 00862 \times 10^{-6}$	$3\cdot 66 \times 10^{-3}$	"	·0297 ∠76° 15′
300 " " " .	3·48	5·86	$\cdot 00893 \times 10^{-6}$	$3\cdot 55 \times 10^{-3}$	"	·0289 ∠80° 13′
400 " " " .	4·01	4·50	$\cdot 00920 \times 10^{-6}$	$3\cdot 44 \times 10^{-3}$	"	·0286 ∠82° 3′
600 " " " .	4·88	2·97	$\cdot 00959 \times 10^{-6}$	$3\cdot 31 \times 10^{-3}$	"	·0284 ∠84° 19′
800 " " " .	—	2·25	$\cdot 00987 \times 10^{-6}$	$3\cdot 22 \times 10^{-3}$	"	·0283 ∠85° 27′
Lead-covered Dry Core Cables:						
Standard cable . .	—	**88**	$\mathbf{\cdot 054 \times 10^{-6}}$	$\mathbf{1\cdot 0 \times 10^{-3}}$	$\mathbf{5 \times 10^{-6}}$	**·154 ∠46° 6′**
Low capacity cable, Spec'n No. 127— 20 lbs. per mile . .	·901	88	$\cdot 054 \times 10^{-6}$	$1\cdot 0 \times 10^{-3}$	5×10^{-6}	·154 ∠46° 6′
Cable to Spec'n No. 132— 6½ lbs. per mile . .	·508	272	$\cdot 0639 \times 10^{-6}$	negligible	" "	·295 ∠44° 33′
Cables to Spec'n No. 125— 10 lbs. per mile . .	·635	176	$\cdot 0714 \times 10^{-6}$	$1\cdot 0 \times 10^{-3}$	" "	·251 ∠45° 24′
20 " " . .	·901	88	" "	" "	" "	·177 ∠46° 13′
40 " " . .	1·27	44	" "	" "	" "	·126 ∠47° 51′
70 " " . .	1·68	26	" "	" "	" "	·0972 ∠50° 3′
100 " " . .	2·01	18	" "	" "	" "	·0816 ∠52° 21′
150 " " . .	2·46	12	" "	" "	" "	·0681 ∠55° 54′
200 " " . .	—	9	" "	" "	" "	·0606 ∠59° 7′

Types of Line for Transmission Calculations.

Units.

Secondary Constants.		Wave Length λ miles.	Wave Velocity W miles per second.	Line Impedance Z_o ohms.	Ratio $\frac{Z_r}{Z_o}$.		
					C.B. Termination.		*L.B.* Instrument.
Attenuation a.	Wave Length β.				Zero local.	300^{ω} local.	
·0373	·0457	137	110,000	1,570 ╲37° 54′	0·266 ╱81° 54′	0·463 ╱67° 54′	0·674 ╱ 97° 54′
·0270	·0382	164	131,000	1,190 ╲35° 45′	0·351 ╱77° 43′	0·612 ╱63° 43′	0·890 ╱ 93° 43′
·0123	·0304	207	165 000	809 ╲20° 40′	0·517 ╱66° 40′	0·902 ╱50° 40′	1·31 ╱ 80° 40′
·00885	·0292	215	171,000	728 ╲15° 27′	0·575 ╱59° 27′	1·00 ╱45° 27′	1·46 ╱ 75° 27′
·00706	·0288	218	174,000	688 ╲12° 26′	0·609 ╱56° 26′	1·06 ╱42° 26′	1·54 ╱ 72° 26′
·00491	·0284	221	176,000	646 ╲ 8° 28′	0·648 ╱52° 28′	1·13 ╱38° 28′	1·64 ╱ 68° 28′
·00396	·0284	221	176,000	622 ╲ 6° 42′	0·672 ╱50° 42′	1·17 ╱36° 42′	1·71 ╱ 66° 42′
·00281	·0282	222	177,000	594 ╲ 4° 30′	0·704 ╱48° 30′	1·23 ╱34° 30′	1·79 ╱ 64° 30′
·00224	·0282	222	177,000	575 ╲ 3° 24′	0·728 ╱47° 24′	1·27 ╱33° 24′	1·85 ╱ 63° 24′
·107	**·111**	**56·6**	**44,900**	**571 ╲42° 50′**	**0·733 ╱86° 50′**	**1·28 ╱72° 50′**	**1·86 ╱102° 50′**
·107	·111	56·6	44,900	571 ╲42° 50′	0·733 ╱86° 50′	1·28 ╱72° 50′	1·86 ╱102° 50′
·210	·207	30·3	24,200	924 ╲44° 33′	0·452 ╱88° 33′	0·790 ╱74° 33′	1·12 ╱104° 33′
·176	·179	35·0	27,900	702 ╲43° 47′	0·596 ╱87° 47′	1·04 ╱73° 47′	1·51 ╱103° 47′
·122	·128	49·0	39,100	497 ╲42° 59′	0·841 ╱86° 50′	1·47 ╱72° 59′	2·14 ╱102° 59′
·0840	·0933	67·2	53,800	352 ╲41° 21′	1·19 ╱85° 21′	2·07 ╱71° 21′	3·01 ╱101° 21′
·0624	·0745	84·4	67,100	273 ╲39° 9′	1·53 ╱83° 9′	2·67 ╱69° 9′	3·89 ╱ 99° 9′
·0499	·0645	97·6	77,500	229 ╲36° 50′	1·84 ╱80° 50′	3·18 ╱66° 50′	4·63 ╱ 96° 50′
·0382	·0564	112·0	88,700	191 ╲33° 17′	2·19 ╱77° 17′	3·82 ╱63° 17′	5·55 ╱ 93° 17′
·0311	·0520	121·0	96,200	170 ╲30° 5′	2·46 ╱74° 5′	4·29 ╱60° 5′	6·24 ╱ 90° 5′

TABLE II.—DATA OF THE MORE IMPORTANT

Metric

Type.	Conductor Weight per kilometre (kilograms).	Primary Constants.				Propagation Constant *P*.
		R ohms.	*C* farads.	*L* henrys.	*S* mhos.	
OPEN WIRES:						
40 lbs. per mile bronze .	11·3	56·0	0·00465 $\times 10^{-6}$	2·61 $\times 10^{-3}$	·621 $\times 10^{-6}$	·0366 /50° 48′
70 „ „ „ .	19·7	32·0	0·00488 $\times 10^{-6}$	2·48 $\times 10^{-3}$	„ „	·0291 /54° 48′
100 „ „ copper .	28·2	10·9	0·00503 $\times 10^{-6}$	2·42 $\times 10^{-3}$	„ „	·0204 /67° 54′
150 „ „ „ .	42·3	7·30	0·00522 $\times 10^{-6}$	2·34 $\times 10^{-3}$	„ „	·0190 /73° 10′
200 „ „ „ .	56·4	5·50	0·00535 $\times 10^{-6}$	2·28 $\times 10^{-3}$	„ „	·0184 /76° 15′
300 „ „ „ .	84·5	3·64	0·00554 $\times 10^{-6}$	2·20 $\times 10^{-3}$	„ „	·0179 /80° 13′
400 „ „ „ .	113	2·79	0·00571 $\times 10^{-6}$	2·14 $\times 10^{-3}$	„ „	·0178 /82° 3′
600 „ „ „ .	169	1·82	0·00595 $\times 10^{-6}$	2·06 $\times 10^{-3}$	„ „	·0176 /84° 19′
800 „ „ „ .	226	1·40	0·00613 $\times 10^{-6}$	2·00 $\times 10^{-3}$	„ „	·0176 /85° 27′
LEAD-COVERED DRY CORE CABLES:						
Standard cable . .	**5·64**	**55·0**	**0·0335 $\times 10^{-6}$**	**·621 $\times 10^{-3}$**	**3·1 $\times 10^{-6}$**	**·0956 /46° 6′**
Low capacity cable, Spec'n No. 127— 20 lbs. per mile .	5·64	55·0	0·0335 $\times 10^{-6}$	·621 $\times 10^{-3}$	3·1 $\times 10^{-6}$	·0956 /46° 6′
Cable to Spec'n No. 132— 6½ lbs. per mile .	1·83	169	0·0396 $\times 10^{-6}$	negligible	„ „	·183 /44° 33′
Cables to Spec'n No. 125— 10 lbs. per mile . .	2·82	109	0·0440 $\times 10^{-6}$	·621 $\times 10^{-3}$	„ „	·156 /45° 24′
20 „ „ . .	5·64	55·0	„ „	„ „	„ „	·110 /46° 13′
40 „ „ . .	11·3	27·0	„ „	„ „	„ „	·0781 /47° 51′
70 „ „ . .	19·7	15·6	„ „	„ „	„ „	·0604 /50° 3′
100 „ „ . .	28·2	10·9	„ „	„ „	„ „	·0507 /52° 21′
150 „ „ . .	42·3	7·30	„ „	„ „	„ „	·0423 /55° 54′
200 „ „ . .	56·4	5·50	„ „	„ „	„ „	·0376 /59° 7′

Types of Line for Transmission Calculations.

Units.

Secondary Constants.		Wave Length λ kilometres.	Wave Velocity W kilometres per second.	Line Impedance Z_o ohms.	Ratio $\frac{Zr}{Z_o}$.		
					C.B. Termination.		*L.B.* Instrument.
Attenuation α.	Wave Length β.				Zero local.	300^ω local.	
·0232	·0284	222	177,000	1,570 ╲37° 54′	0·266 ╱81° 54′	0·463 ╱67° 54′	0·674 ╱ 97° 54′
·0168	·0238	264	210,000	1,190 ╲33° 43′	0·351 ╱77° 43′	0·612 ╱63° 43′	0·890 ╱ 93° 43′
·00764	·0189	334	265,000	809 ╲20° 40′	0·517 ╱66° 40′	0·902 ╱50° 40′	1·31 ╱ 80° 40′
·00549	·0182	348	276,000	728 ╲15° 27′	0·575 ╱59° 27′	1·00 ╱45° 27′	1·46 ╱ 75° 27′
·00438	·0179	351	280,000	688 ╲12° 26′	0·609 ╱56° 26′	1·06 ╱42° 26′	1·54 ╱ 72° 26
·00304	·0176	356	283,000	646 ╲ 8° 28′	0·648 ╱52° 28′	1·13 ╱38° 28′	1·64 ╱ 68° 28′
·00246	·0176	356	283,000	622 ╲ 6° 42′	0·672 ╱50° 42′	1·17 ╱36° 42′	1·76 ╱ 66° 42′
·00175	·0175	359	285,000	594 ╲ 4° 30′	0·704 ╱48° 30′	1·23 ╱34° 30′	1·79 ╱ 64° 30′
·00139	·0175	359	285,000	575 ╲ 3° 24′	0·728 ╱47° 24′	1·27 ╱33° 24′	1·85 ╱ 63° 24′
·0663	**·0689**	**91·1**	**72,300**	**571 ╲42° 50′**	**0·733 ╱86° 50′**	**1·28 ╱72° 50′**	**1·86 ╱102° 50′**
·0663	·0689	91·1	72,300	571 ╲42° 50′	0·733 ╱86° 50′	1·28 ╱72° 50′	1·86 ╱102° 50′
·131	·128	48·8	39,000	924 ╲44° 33′	0·452 ╱88° 33′	0·790 ╱74° 33′	1·12 ╱104° 33′
·109	·112	56·4	45,000	702 ╲43° 47′	0·596 ╱87° 47′	1·04 ╱73° 47′	1·51 ╱103° 47′
·0758	·0794	78·9	63,000	497 ╲42° 59′	0·841 ╱86° 59′	1·47 ╱72° 59′	2·14 ╱102° 59′
·0524	·0579	108	86,700	352 ╲41° 21′	1·19 ╱85° 21′	2·07 ╱71° 21′	3·01 ╱101° 21′
·0388	·0462	136	108,000	273 ╲39° 9′	1·53 ╱83° 9′	2·67 ╱69° 9′	3·89 ╱ 99° 9′
·0310	·0401	157	125,000	229 ╲36° 50′	1·84 ╱80° 50′	3·18 ╱66° 50′	4·63 ╱ 96° 50′
·0237	·0351	180	143,000	191 ╲33° 17′	2·19 ╱77° 17′	3·82 ╱63° 17′	5·55 ╱ 93° 17′
·0193	·0323	195	155,000	170 ╲30° 5′	2·46 ╱74° 5′	4·29 ╱60° 5′	6·24 ╱ 90° 5′

Table III.—Data of the less Important Types of Line for Transmission Calculations.

British Units.

Type.	Primary Constants.				Secondary Constant.
	R ohms.	*C* farads.	*L* henrys.	*S* mhos.	Attenuation *a*.
Lead - covered Dry Core Cables:					
Cables to Spec'n No. 126—					
20 lbs. per mile	88	$\cdot 0822 \times 10^{-6}$	$1 \cdot 0 \times 10^{-3}$	5×10^{-6}	·131
40 ,, ,, ...	44	,, ,,	,, ,,	,, ,,	·0905
70 ,, ,, ...	26	,, ,,	,, ,,	,, ,,	·0669
100 ,, ,, ...	18	,, ,,	,, ,,	,, ,,	·0534
150 ,, ,, ...	12	,, ,,	,, ,,	,, ,,	·0409
200 ,, ,, ...	9	,, ,,	,, ,,	,, ,,	·0333
Cable to Spec'n No. 10—					
12½ lbs. per mile . . .	144	$0 \cdot 54 \times 10^{-6}$	,, ,,	,, ,,	·138
Rubber-covered Dry Core Aerial Cables:					
Spec'n No. 134—					
6½ lbs. per mile . . .	272	$\cdot 0785 \times 10^{-6}$	negligible	,, ,,	·232
Special, weight under 1 lb. per foot—					
6½ lbs. per mile . . .	272	$\cdot 0987 \times 10^{-6}$	,,	,, ,,	·260
Spec'n No. 130—					
10 lbs. per mile . . .	176	$\cdot 0775 \times 10^{-6}$	$1 \cdot 0 \times 10^{-3}$	,, ,,	·183
Spec'n No. 20A—					
12½ lbs. per mile . . .	144	$\cdot 0700 \times 10^{-6}$	,, ,,	,, ,,	·157
Spec'n Nos. 20 and 131—					
20 lbs. per mile . . .	88	$\cdot 0700 \times 10^{-6}$	,, ,,	,, ,,	·122
Miscellaneous Wires and Cables:					
22/15 *V.I.R.* opening-out . .	146	$\cdot 250 \times 10^{-6}$	$1 \cdot 3 \times 10^{-3}$	$\frac{1}{\text{infinity}}$	·297
20/12 twin *V.I.R.*	87	$\cdot 225 \times 10^{-6}$	,, ,,	,,	·213
20/10 *V.I.R.* cable, with steel suspender	87	$\cdot 300 \times 10^{-6}$	,, ,,	,,	·246
20/10 twin *V.I.R.* leading-in and opening-out	87	$\cdot 200 \times 10^{-6}$	,, ,,	,,	·201
Silk and cotton cable—					
9¼ lbs. per mile . . .	192	$\cdot 100 \times 10^{-6}$	negligible	,,	·219

TABLE IV.—DATA OF THE LESS IMPORTANT TYPES OF LINE FOR TRANSMISSION CALCULATIONS.

Metric Units.

Type.	Conductor Weight per kilometre (kilograms).	Primary Constants.				Secondary Constant.
		R ohms.	*C* farads.	*L* henrys.	*S* mhos.	Attenuation *α*.
LEAD-COVERED DRY CORE CABLES:						
Cables to Spec'n No. 126—						
20 lbs. per mile . .	5·64	55·0	$0·0510 \times 10^{-6}$	$·621 \times 10^{-3}$	$3·1 \times 10^{-6}$	·0314
40 ,, ,, . .	11·3	27 0	,, ,,	,, ,,	,, ,,	·0562
70 ,, ,, . .	19·7	15·6	,, ,,	,, ,,	,, ,,	·0415
100 ,, ,, . .	28·2	10·9	,, ,,	,, ,,	,, ,,	·0332
150 ,, ,, . .	42 3	7·30	,, ,,	,, ,,	,, ,,	·0254
200 ,, ,, . .	56·4	5·50	,, ,,	,, ,,	,, ,,	·0207
Cable to Spec'n No. 10—						
12½ lbs. per mile . .	3·52	89·0	$0·054 \times 10^{-6}$	,, ,,	,, ,,	·0857
RUBBER-COVERED DRY CORE AERIAL CABLES:						
Spec'n No. 134—						
6½ lbs. per mile . .	1·83	169	$0·0187 \times 10^{-6}$	negligible	,, ,,	·144
Special, weight under 1 lb. per foot—						
6½ lbs. per mile . .	1·83	169	$0·0613 \times 10^{-6}$	,,	,, ,,	·161
Spec'n No. 130—						
10 lbs per mile . .	2·82	109	$0·0181 \times 10^{-6}$	$·621 \times 10^{-3}$	,, ,,	·114
Spec'n No. 20A—						
12½ lbs. per mile . .	3 52	89·0	$0·0435 \times 10^{-6}$	,, ,,	,, ,,	·0975
Spec'n Nos. 20 and 131—						
20 lbs. per mile . .	5·64	55·0	$0·0435 \times 10^{-6}$	,, ,,	,, ,,	·0758
MISCELLANEOUS WIRES AND CABLES:						
22/15 *V.I.R.* opening-out .	3·40	91·0	$0·155 \times 10^{-6}$	$·808 \times 10^{-3}$	$\frac{1}{\text{infinity}}$	·184
20/12 twin *V.I.R.* . .	5·70	54·0	$0·140 \times 10^{-6}$	,, ,,	,,	·132
20/10 *V.I.R.* cable, with steel suspender . .	5·70	54·0	$0·186 \times 10^{-6}$	,, ,,	,,	·153
20/10 twin *V.I.R.* leading-in and opening-out .	5·70	54·0	$0·124 \times 10^{-6}$	,, ,,	,,	·125
Silk and cotton cable—						
9¼ lbs. per mile . .	2·60	119	$0·0620 \times 10^{-6}$	negligible	,,	·136

TABLE V.—TRANSMISSION EQUIVALENTS.

Type.	Transmission Equivalent.	Reciprocal of Equivalent.
OPEN WIRES :		
40 lbs. per mile bronze	2·830	0·353
70 ,, ,, ,,	3·890	0·257
100 ,, ,, copper	8·440	0·118
150 ,, ,, ,,	11·680	0·0853
200 ,, ,, ,,	14·710	0·0680
300 ,, ,, ,,	21·000	0·0476
400 ,, ,, ,,	26·050	0·0384
600 ,, ,, ,,	36·750	0·0272
800 ,, ,, ,,	45·750	0·0218
LEAD-COVERED DRY CORE CABLES :		
Standard cable	**1·000**	**1·000**
Low capacity cable, Spec'n No. 127—		
20 lbs. per mile	1·000	1·000
Cable to Spec'n No. 132—		
6½ lbs. per mile	0·509	1·965
Cables to Spec'n No. 125—		
10 lbs. per mile	0·605	1·654
20 ,, ,,	0·872	1·147
40 ,, ,,	1·262	0·792
70 ,, ,,	1·705	0·587
100 ,, ,,	2·130	0·470
150 ,, ,,	2·775	0·360
200 ,, ,,	3·400	0·294
Cables to Spec'n No. 126—		
20 lbs. per mile	0·810	1·235
40 ,, ,,	1·175	0·850
70 ,, ,,	1·590	0·629
100 ,, ,,	1·990	0·502
LEAD-COVERED DRY CORE CABLES (*continued*) :		
Cables to Spec'n No. 126 (*continued*)—		
150 lbs. per mile	2·588	0·386
200 ,, ,,	3·168	0·316
Cable to Spec'n No. 10—		
12½ lbs. per mile	0·775	1·290
RUBBER-COVERED DRY CORE AERIAL CABLES :		
Spec'n No. 134—		
6½ lbs. per mile	0·460	2·173
Special, weight under 1 lb. per foot—		
6½ lbs. per mile	0·410	2·440
Spec'n No. 130—		
10 lbs. per mile	0·582	1·718
Spec'n No. 20A—		
12½ lbs. per mile	0·678	1·475
Spec'n Nos. 20 and 131—		
20 lbs. per mile	0·880	1·136
MISCELLANEOUS WIRES AND CABLES :		
22/15 *V.I.R.* opening out	0·359	2·785
20/12 twin *V.I.R.*	0·497	2·010
20/10 *V.I.R.* cable, with steel suspender	0·430	2·325
20/10 twin *V.I.R.* leading-in and opening-out	0·528	1·892
Silk and cotton cable, 9¼ lbs. per mile	0·486	2·058

CHAPTER IX

LOADED CABLES IN PRACTICE

1. Modern Improvements in Telephonic Cables and Lines.—The result of nearly twenty years' investigations by mathematical physicists and practical telephonists, starting from the date of Mr. Oliver Heaviside's first fertile suggestions, has been to effect a great improvement in the transmitting powers of telephonic lines by working in the direction indicated by Heaviside, viz., that an increase in the inductance of the line would reduce attenuation and distorsion. Although many schemes were put forward for increasing the inductance of the line by enclosing it in iron, and several alternative proposals, such as those of Professor S. P. Thompson, for placing across it inductive shunts, it cannot be said that the suggestions bore much practical fruit until after Professor Pupin's important contribution to the subject by his proposal to locate the inductance in equispaced loading coils, coupled with a practical rule for their effective spacing. The result of this has been that practical experience has now accumulated to a considerable extent in connection with the two methods of carrying out the Heaviside-Pupin recommendations, viz., increasing the inductance of the line by uniform loading and increasing it by loading coils at intervals.

The uniform loading consists in wrapping or enclosing the copper conductor in iron wire in such a manner that the magnetic flux produced around it by the telephonic currents is increased, with a corresponding increase in the effective inductance, and therefore diminution of the attenuation constant, with more or less reduction in the distorsion of the wave form produced by the line.

Three cases present themselves for consideration, viz., aerial or overhead lines, underground cables, and submarine telephonic cables. We shall describe briefly what has been attempted and achieved in each case. The improvement of telephony conducted through overhead or aerial conductors has been effected solely through the use of loading coils. Aerial lines are not adapted for uniform loading. It would involve a great increase in the weight per mile and necessitate stronger cables and more expensive supports, and also offer greater surface to wind and snow. The writer is not aware that it has ever been tried. On the other hand, aerial lines are well suited for loading coils, since these can be attached at intervals to the posts which carry the line.

So far, then, uniform loading has been restricted to underground cables and to submarine cables, whilst the non-uniform loading or application of loading coils has been extensively tried on underground lines, and in a few cases, but with great success, in the case of under-water cables.

In respect, however, of the improvement gained or to be gained in the case of aerial lines and underground or underwater cables respectively, the following remarks of Dr. Hammond V. Hayes in a paper read before the St. Louis International Electrical Congress are important[1]:

"In the case of cables there is a distinct improvement in the "quality of the transmission produced by the introduction of "the loading coils, the voice of the speaker being received more "distinctly. The high insulation which can be maintained at "all times on cable circuits renders it possible to introduce "loading coils upon the circuits without danger of materially "augmenting leakage losses. The marked diminution in "attenuation, the improvement in quality of transmission, and "the ease with which inductance coils can be placed on cable "circuits without introducing other injurious factors, such as "leakage or cross-talk with other circuits, renders the use of "loaded cable circuits especially attractive."

"The reduction of attenuation that can be obtained by the "introduction of loading coils on air-line circuits, even under

[1] See reprint of this paper in *The Electrician*, Vol. LIV., p. 362, December 16th, 1904, "Loaded Telephone Lines in Practice."

"theoretically perfect conditions, is less than can be obtained on "cable circuits. This difference in the effectiveness of loading "between the two classes of circuits, as far as attenuation is con- "cerned, can be explained by the fact that on a cable circuit the "capacity is large and the inductance of the circuit itself is "practically negligible, due to the proximity of the two wires of "the pair. On aerial circuits, on the other hand, the distance "between the outgoing and return wire is such as to make the "capacity of the circuit much less, and its inductance much "greater. This larger self-induction of the open-wire circuit "operates to decrease the attenuation, and, as it were, to rob the "loading coils of part of their usefulness. Again, the insulation "of an aerial circuit cannot be maintained as high as that of a "cable circuit, so that the added inductance due to the intro- "duction of loading coils upon the line tends to increase the "losses due to leakage."

"Moreover, there is not the same improvement in the quality "of transmission on a loaded aerial circuit, as compared with a "similar circuit unloaded, as is found between loaded and "unloaded cables. Initially, open-wire circuits are practically "free from distorsion, whereas the distorsion on cable circuits of "long length is considerable. The addition, therefore, of loading "coils to aerial circuits cannot be expected to effect any improve- "ment in the quality of transmission, whereas in the case of "cables the introduction of the additional inductance renders "the circuits practically distorsionless and effects a marked "improvement in the clearness of the transmitted speech."

It is perhaps well to point out here that the two qualities essential in telephonically transmitted speech are sufficient *loudness* or volume of sound and *clearness* or distinctness. Both these qualities are necessary for intelligibility. There may be clearness, but the speech may be so faint that only people with exceptionally good hearing can comprehend it. On the other hand, there may be loudness but not clearness, and the speech is then also not intelligible. The loss of volume is due to the attenuation generally, but the loss of distinctness to the difference in the attenuation of the different harmonic frequencies and consequent distorsion of the wave form.

In the case of the aerial lines the want of loudness in the transmitted sound is chiefly due to the resistance of the line, and in so far as this is the cause it cannot be much alleviated by the introduction of inductance. It is only the attenuation which arises from distributed capacity which can be reduced by added inductance. In cables, on the other hand, the predominant cause of the attenuation is, generally speaking, capacity, and it is therefore appropriately remedied by the introduction of inductance.

Nevertheless experience shows that some advantage is gained by the introduction of loading coils into aerial lines.

2. The Introduction of Loading Coils into Overhead or Aerial Lines.—The effect of introducing inductance coils of low resistance into aerial lines has now been

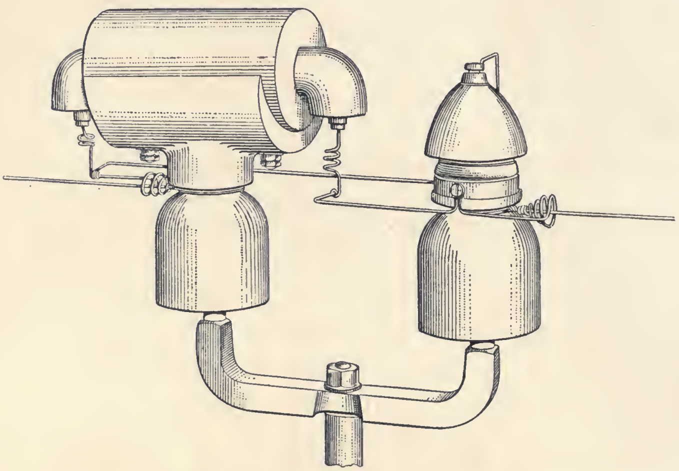

FIG. 1.—Loading Coil used in the Berlin-Magdeburg Aerial Telephone Line.

tried on several long lines, and found to be an advantage. These coils take the form of a closed iron circuit-choking coil having a laminated or iron wire core, covered over with a low resistance wire.

The general form of coil and core and leading-in sleeve may be seen from the diagram in Fig. 1, which represents the coils used on the first German line so treated, viz., the Berlin-Magdeburg line, 150 km. in length. The coils were mounted on an arm together with a vacuum lightning arrester, mounted in parallel with the coil.

After a preliminary trial on the Berlin-Magdeburg line it was decided to equip a longer line, and the Berlin-Frankfort-on-Main was chosen, as the distance is about 580 km. (= 360 miles). A new bronze wire, 2·5 mm. in diameter, was accordingly run. Also between the terminal points there existed two other bronze wires, one 4 mm. in diameter and the other 5 mm. All lines were double wire lines. The inductance or loading coils were inserted every 5 km. on the 2·5-mm. line. The effective resistance of each coil was 8·7 ohms, and its inductance 0·11 henry. Hence the coils add 3·48 ohms to the resistance, and 0·044 henry to the inductance per kilometre of loop or distance. The general result as regards speech transmission was that, whereas before loading the speech volume on the 2·5-mm. line was of course less than that on the 5-mm. and 4-mm. lines, after loading the loaded 2·5-mm. line was better than the 4-mm. unloaded line, but not quite so good as the 5-mm. unloaded line.

The following are the constants and attenuation constants of these four lines at a frequency of 900:

LINE.	Resistance R in ohms.	Inductance L in henrys.	Capacity C in microfarads.	Attenuation Constant α.
Bronze wire 5 mm. diameter unloaded	1·92	0·00186	0·0063	0·00176
Ditto 4 mm. diameter	3·00	0·00194	0·0060	0·00262
Ditto 2·5 mm. diameter unloaded	7·70	0·00214	0·0055	0·00591
2·5 mm. diameter loaded every 5 km.	11·18	0·0461	0·0055	0·00193

R is the effective resistance in ohms per kilometre of loop; L is the inductance in henrys per kilometre of loop; C is the capacity wire to wire in microfarads per kilometre of loop; a is the attenuation constant per kilometre of loop.

The loaded 2·5-mm. line is equivalent to an unloaded 4·7-mm. line of the same material.

The product of the attenuation constant and the length of the line, called the *attenuation length*, is as follows:

1. For the 5-mm. line $al = 0{\cdot}95$,
2. For the 4-mm. line $al = 1{\cdot}52$,
3. For the 2·5-mm. line unloaded $al = 3{\cdot}43$,
4. For the 2·5-mm. line loaded $al = 1{\cdot}12$.

The smaller the attenuation length al the better the speech-transmitting qualities of the line. It is generally considered that a line permits excellent talking when al is not more than 2·5, and fair speech when al does not exceed 3·5. Hence the 2·5-mm. unloaded line is efficient, but becomes better on loading.

It has been agreed that with an ordinary copper line joined directly to the telephonic apparatus the relation between speech and attenuation length al is as follows:

Speech	*up to*	*Attenuation lengths al. equal to*
Very good	,, ,,	2·5
Good	,, ,,	3·5
Practical limit	*at*	4·8

This corresponds to about forty-six miles of the National Telephone Company's standard cable when using the standard type of central battery instrument and circuit at either end of the line, and a subscribers' line of 300-ohms resistance.

The result therefore of loading, in the above manner, the Berlin-Frankfort-on-Main 2·5-mm. line has been to effect a sensible increase in the speech efficiency of the line.

Previously to the equipment of the above long distance line experiments had been tried on the Berlin-Magdeburg overhead line, 2-mm. bronze wire, 150 km. in length.

This line was equipped with loading coils having an effective resistance of six ohms and an inductance of 0·08 henry placed every 4 km. The result was better speech than that over a 3-mm. bronze wire 180 km. in length running between the same places.

Also between Berlin and Potsdam (32·5 km.) on certain lines coils of 4·1 ohms and 0·062 henry were introduced every 1·3 km. The result was an increase in the inductance per km. of two hundredfold and a reduction of the attenuation constant to one-sixth of that of the unloaded line.

In loading an aerial line or a cable it is, however, necessary to make arrangements to avoid losses by reflection at the point where the loaded line joins on to an unloaded or terminal line. It has already been explained that when a telephone wave passes across the junction of two lines which differ considerably in inductance or capacity per unit of length there is a reflection of energy which acts to produce an increased attenuation in certain cases. In practice the effect of reflection is very considerable, particularly when the loaded section is relatively not long. Theoretically this reflection can be eliminated by the introduction of a perfect transformer at every point of discontinuity in the line; practically it is best overcome by the employment of what is called a *terminal taper*. This consists in a series of several inductance coils placed near the ends of the loaded section, each one having somewhat less inductance than the preceding one and less than that of the coils in the main loaded section. Hence the inductance per mile or per kilometre is not suddenly changed, but reduced gradually or tapered off from that in the loaded section to that in an unloaded line. The spacing of the coils in the taper is the same as that in the

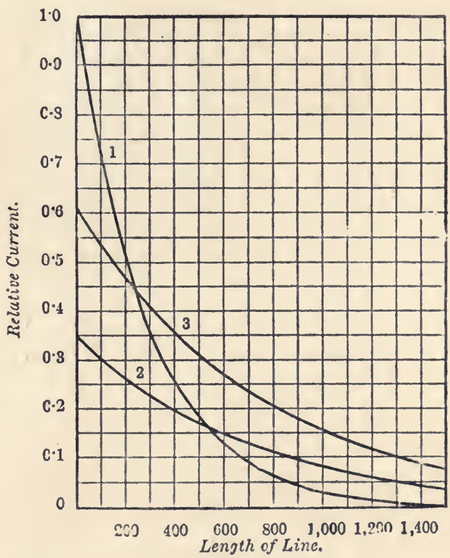

FIG. 2.—Curves showing effect of loading Coils on an aerial line 435 lbs. to the mile (H. V. Hayes).

main part of the loaded line. This taper is introduced at both ends.

The effect of taper and loading is well shown in some curves which have been given by Dr. Hammond V. Hayes in an interesting paper[1] entitled "Loaded Telephone Lines in Practice." The coils used were toroidal in shape, about 10 inches in diameter and 4 inches high, and had an effective resistance of 15·5 ohms at 2,000 periods per second, but only 2·4 ohms steady resistance and an inductance of 0·25 henry. On aerial circuits such coils are placed about two miles or so apart, so as to give an inductance of about 0·1 henry per mile.

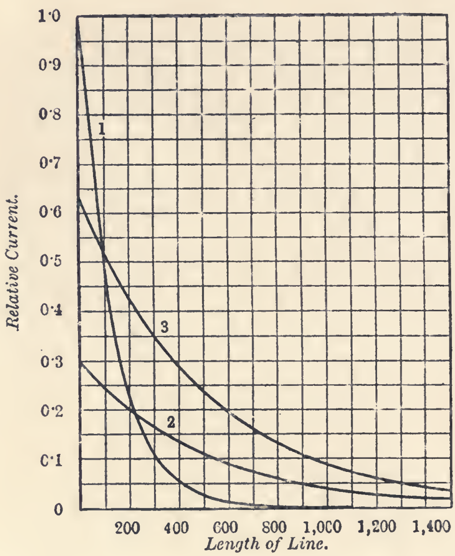

FIG. 3.—Curves showing effect of loading on an aerial line 176 lbs. to the mile (H. V. Hayes).

The curves in Fig. 2 show the effect of such loading on an aerial line weighing 435 lbs. to the mile. Curve 1 shows the decrease in current at the receiving end for various lengths when the line is unloaded, curve 2 when the transmitting and receiving instruments are connected to the loaded line without taper, and curve 3 the same when the line is tapered at both ends.

The curves in Fig. 3 show the same results, but for a line consisting of wire 176 lbs. per mile, and, as before, curve 1 shows the attenuation of the unloaded line, curve 2 of the loaded untapered line, and curve 3 the loaded and tapered line. These curves show clearly that for short lengths of line loading is not

[1] Read before Section 6 of the St. Louis International Electrical Congress, 1904; also see *The Electrician*, December 16th, 1904, Vol. LIV., p. 362, or *Science Abstracts*, VII. B, Abs. 2,968, 1904.

beneficial, but, on the contrary, reduces the received current considerably. This is because the added resistance increases the attenuation constant at first more than the added inductance reduces it.

3. Loaded Underground Cables.—As already remarked, the benefits to be expected from loading a line either continuously or at intervals are likely to be more pronounced in the case of cables than of aerial lines, for the reason that the capacity per mile is always greater in the case of cables, and therefore its peculiar effect in producing attenuation and distorsion is capable of remedy by suitably introduced inductance.

Moreover, in underground cables there are no particular difficulties involved in introducing the inductance coils when spaced impedance is added. The coils can be of any convenient size and can be located in small watertight chambers placed at regular intervals on the line.

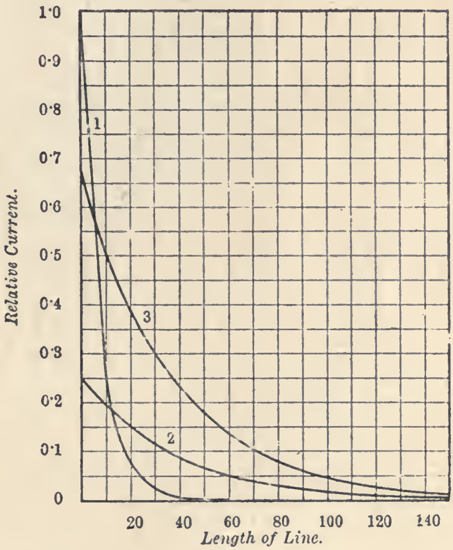

FIG. 4.—Curves showing effect of loading on a Telephone Cable (H. V. Hayes).

Dr. Hammond V. Hayes has given in the same paper (*loc. cit.*) some curves for loaded cables similar to those above given for aerial lines.

Fig. 4 shows the result of loading a telephone cable having a pair of wires each 0·03589-inch diameter and a resistance of 96 ohms per mile of circuit (double wire circuit). The capacity is 0·068 microfarad per mile. The inductance added by the loading coils amounted to about 0·6 henry per mile.

Curve 1 in Fig. 4 shows the attenuation on the unloaded cable, curve 2 the same for the loaded cable without taper, and curve 3 the attenuation for the loaded and tapered line. It will be seen that the effect of loading without taper is to reduce greatly the sending end current and to increase the received current beyond a certain length of line.

The effect of loading with taper is to reduce somewhat the sending end current, but to greatly increase the received current beyond short distances when compared with the unloaded line.

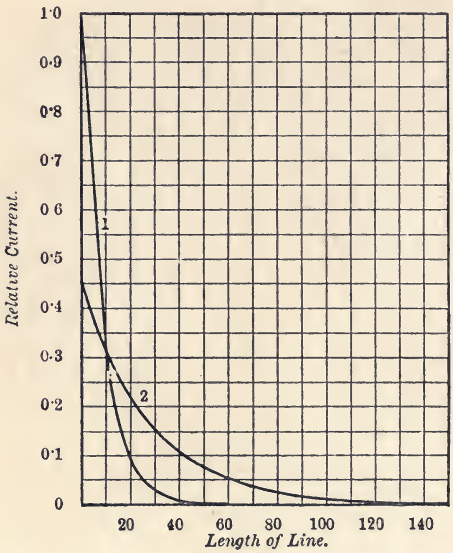

FIG. 5.—Curves showing the effect of loading on a Telephone Cable (H. V. Hayes).

A comparison of curves 2 and 3 shows how great a factor the reflection losses are between the terminal apparatus and the loaded line and how important it is to employ taper to reduce these losses. In Fig. 5 are given two curves. Curve 1 is the attenuation curve of an unloaded line, and curve 2 for the same line lightly loaded and without taper. It is seen that the reflection losses are much reduced, and that when no taper is employed it is easily possible to overload the line detrimentally.

The results of loading as far as the cable itself is concerned can be predicted by means of the formulæ given, but it is less easy to foresee the exact results when tapering is not employed. Hence in those numerous cases in which a loaded trunk cable has aerial lines connected on at both ends the importance of introducing suitable taper is very great.

The necessity for maintaining good insulation on loaded cables is discussed in a later section of this chapter. Meanwhile it may be stated that loaded underground cables have been extensively employed by the National Telephone Company in Great Britain with great advantage.

The type of impedance coil adopted after careful experiment is

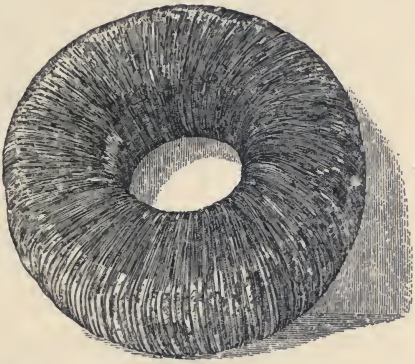

FIG. 6.—Loading or Inductance Coil (without case) as used by the National Telephone Company of Great Britain.

shown in Fig. 6. It consists of a choking coil having a closed magnetic circuit formed of fine soft iron wire and overlaid with silk-covered insulated copper wire. The finished toroidal coil has an overall diameter of about 4·5 to 5 inches, and a central aperture of about 1·5 inches, and a depth of nearly 2 inches.

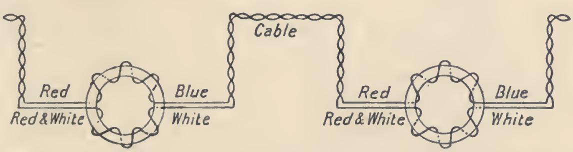

FIG. 7.—A Diagram showing the mode of Winding the Loading Coil in two parts and their insertion in the two sides of the Cable.

The effective resistance of such a coil may be from 3·5 to 15 ohms for currents of 1,000 frequency, and the inductance may be from 0·06 to 0·25 henry. Each coil is wound in two parts, one-half being inserted in the lead and one in the return (see Fig. 7).

The following table gives the data of some of the coils employed:

DATA FOR LOADING COILS.

Loading.	Spacing Interval in miles between coils.	Steady Resistance in ohms.	Effective Resistance in ohms at a frequency of 1,000.	Inductance in henrys.
Very light . . .	5·75	1·18	3·5	·059
Light	2·5	2·84	7·5	·133
Medium . . .	1·75	3·97	11·7	·176
Heavy . . .	1·25	6·11	15·7	·252

The toroidal coils are enclosed in a watertight iron case, and

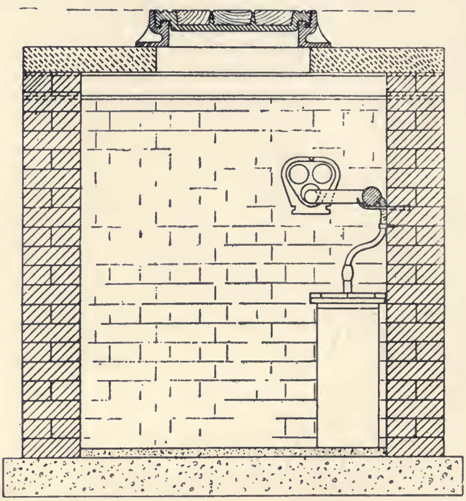

FIG. 8.—A Brick Pit for containing an Iron Case in which are a number of Loading Coils, as constructed by the National Telephone Company.

a number of these coils may be placed in a brick pit and inserted in the circuit of cables passing through the pit (see Fig. 8).

4. Loaded Submarine or Under-water Telephone Cables.—Whilst there is little or no difficulty in introducing loading coils into aerial lines or underground cables, the problem of applying these methods to under-water cables presents peculiar difficulties. Any considerable enlargements on a submarine cable must not only add to its weight and to the strains experienced during laying, but may also increase the difficulties of laying very greatly. It was therefore with some hesitation that telegraphic engineers approached this particular work, and it was only when the great and certain improvements made by loading land lines had clearly established beyond doubt that submarine telephony must be equally improved, if the mechanical difficulties of making and laying such a cable could be overcome, that the matter was taken seriously in hand. Even then it was felt that the difficulties of manufacture and laying of a continuously loaded cable might be less than those of a loaded cable, and the first efforts seem to have been in this direction.

The continuously loaded cable has, however, two disadvantages as compared with the non-uniformly loaded cable. It is undoubtedly more expensive to make, and it is not possible to predict with any certainty the attenuation constant of a cable so made. This arises from the impossibility of determining beforehand the permeability of the iron wire which is laid over the core to increase its permeability, and also from changes in that permeability during and after laying, and also from the unknown increase in the effective resistance of the core which results from the iron wire envelope due to hysteresis and eddy currents.

The general construction of a continuously loaded cable is as follows: The copper core is insulated and overlaid with several windings of fine iron wire, and this is insulated either with gutta-percha or with paper. If the latter is used, then a continuous lead covering has to be put over the paper to keep it dry, and over that protecting layers of jute or hemp and then the usual steel armouring. The iron wire laid over the copper then increases the inductance to a certain extent not easy to foretell accurately. Cables on this plan have been laid in Germany and Holland, and the following details and table are taken from a valuable paper by Major O'Meara, C.M.G., Engineer-in-chief of the British Postal Telegraphs, read before the Institution of Electrical Engineers of London in November, 1910, "On Submarine Cables for Long-distance Telephone Circuits."

Major O'Meara states that the first continuously loaded cable having the copper conductor wrapped with a layer of 0·008-inch iron wire on the plan devised by Mr. C. E. Krarup, the Engineer-in-chief of the Danish Telegraph Service, appears to have been that laid by the Danish Government, in November, 1902, between Elsinore and Helsingborg.[1] Mechanical and electrical data of this cable are given in the table. The dielectric was gutta-percha, and, except in respect of the iron wrapping, the cable did not differ materially from the ordinary type of submarine cable. This was followed, as will be seen from the table, by various paper-insulated cables having the conductors wrapped with a single layer of 0·012-inch iron wire. The cable distinguished by the letter E in the table on p. 278, was laid in July, 1904. Each copper conductor consists of a central wire about 0·089 inch in diameter surrounded by three copper strips each 0·094 inch wide and 0·020 inch thick. The sectional area of the copper is approximately 0·0124 square inch, and the weight per knot 285 lbs. The iron wrapping consists of three layers of 0·008-inch wire, and the insulator is gutta-percha having an external diameter of 0·354 inch. The four cores are laid up with an inner serving of tanned jute and an outer serving of tarred jute yarn to a diameter of 1·18 inch, and sheathed with fifteen galvanised iron wires of roughly trapezoidal section. The external covering appears to be the usual tarred yarn and compound.

The electrical constants of the cable per knot from Mr. Krarup's figures are given on p. 277.

Of the paper-insulated lead-covered cables the Dano-German telephone cable laid between Fehmarn and Lolland in 1907 may be taken as representative. The copper conductor with its triple soft iron wire wrapping is precisely similar to that used in the Seeland-Samso-Jutland cable described above. The insulator consists of paper cord laid on in an open spiral followed by a

[1] "Moderne Telefonkabler," by C. E. Krarup, *Elektroteknikeren*, December 10th, 1904.

close wrapping of paper ribbon up to a diameter of 0·303 inch. Four of the cores so formed are stranded together with the necessary worming and then covered with paper to a diameter of 0·787 inch. The diagonal distance apart of the cores, centre to centre, is 0·413 inch. The core after being thoroughly dried is next sheathed with two layers of lead alloyed with 3 per cent. of tin, each layer being 0·055 inch thick. The lead sheath is seamless, watertight, and continuous throughout the entire length of the core. Outside the lead sheath is a double layer of asphalted paper and a layer of jute and compound. The armour consists of thirteen galvanised iron wires or strips of trapezoidal section $\left(\frac{0{\cdot}315+0{\cdot}252}{2} \times 0{\cdot}157 \text{ square inch}\right)$, and over this is a double layer of jute and compound.

Resistance. Ohms per Knot of Conductor.		Capacity. Microfarads per Knot of Conductor.		Inductance. Knots per Millihenry.	
Steady Current.	Alternating Current, $n=900$.	Steady Current.	Alternating Current.	With Iron.	Without Iron.
3·971	4·175	0·4983	0·4454	8·07	0·93

To prevent the destruction of the cable by the puncture of the lead sheath at any point, solid plugs 1 metre (3·28 feet) long are inserted at every 150 metres (164 yards). The constants of the cable are as follows :

Resistance per knot of loop, 8·924 ohms ... } Continuous
Capacity per knot of loop, 0·0872 microfarad... } current.

Capacity per knot of loop, 0·0770 microfarad... { Alternating current.

Inductance per knot of loop, 18·26 to 18·09 millihenrys.

The table on p. 278, taken by permission from Major O'Meara's paper (*loc. cit.*), gives the details of some continuously loaded cables.

Some Foreign "Continuously Loaded" Submarine Cables.

	Cable.	Date of laying.	Length in Knots.	Number of Conductors.	Copper Square Inch.	Wrapping of Iron Wire.	Insulation Thickness in Inches.	Lead Sheath.	Resistance in Ohms per Knot of Conductor.		Mutual Capacity Farads per Knot of Conductor.		Self-induction Millihenrys per Knot of Conductor.		Old Formula (Preece) $C' \gamma$.	Attenuation Constant per Knot Frequency =900.
									γ Continuous Current.	ω Alternating Current $n=(900)$.	C' Continuous Current.	C Alternating Current.	Prepared, wrapped with Iron Wire.	Without Iron Wire.		
						Mils.										
A	Elsinore-Helsingborg .	Nov., 1902	2·85	4	0·0054	7·88	Gutta-percha 0·323	No	8·49	8·79	$0·324 \times 10^{-6}$	$0·3030 \times 10^{-6}$	4·92	1·100	$2·750 \times 10^{-6}$	0·0340
B	Fahmarn-Laaland . .	Jan., 1903	10·38	4	0·0155	11·82	Impregnated paper	Yes	3·18	4·79	0·301 ,,	0·2660 ,,	4·65	0·850	1·075 ,,	0·0184
C	Greetsiel (Emden)-Borkum	Spring, 1903	15·85	4	0·0054	11·82	Air-space paper	Yes	9·03	11·08	0·138 ,,	(0·1243) ,,	7·41	1·240	1·335 ,,	0·0225
D	Cuxhaven-Heligoland .	Autumn, 1903	40·50	2+2/2	0·0186	11·82	Solid paper	Yes	2·53	3·38	0·170 ,,	(0·1490) ,,	3·98	0·595	0·427 ,,	0·0104
E	Seeland-Samso-Jutland .	July, 1904	8·93+11·02	4	0·0124	3×7·88	Gutta-percha 0·355	No	3·98	4·18	0·500 ,,	0·4460 ,,	8·08	(0·900)	1·970 ,,	0·0156

The figures in brackets are, however, interpolated or approximately stated.

A great difference seems to exist between the attenuation constants of continuously loaded cables as actually measured and that which theory would predict so far as the measured data allow. Thus for a cable made for the Danish Telegraph Service continuously loaded with three layers of iron wire 0·0079-inch diameter the observed attenuation constant was 0·0296, whereas that calculated from certain data as to the permeability of the iron was 0·0197.

An additional objection to continuous loading by envelopes of iron wire is that it increases the capacity by increasing the diameter of the conductor. Also, in the opinion of experts, its cost is about twice as great as that of Pupin loading for equal effect.

Accordingly attention has more recently been directed to the question of designing under-water cables loaded with Pupin coils at intervals, and two successful examples of this are the loaded lead-covered telephone cable laid by Messrs. Siemens and Halske across Lake Constance in 1906 and the loaded submarine telephone cable laid by the British Postal Telegraph Department in 1910 across the English Channel between Abbotscliff, in England, and Grisnez, in France.

The Lake Constance cable is about $9\frac{1}{3}$ miles long. The maximum depth of the cable is 138 fathoms, at which depth the pressure is about 25 atmospheres. The cable contains seven speaking circuits, and these cores are enclosed in insulation and covered with steel armour, over which a continuous coating of lead-tin alloy is pressed and then a jute covering and a second armour. The loading coils are cylindrical and are slipped over the cable and connected in circuit with one conductor. The capacity wire to wire is 0·038 microfarad per kilometre, and the inductance, including loading coils, is 0·20 henry, and the effective resistance is 33·5 ohms per kilometre at a frequency of 900. The attenuation constant is 0·0072 per kilometre. For details of the work of laying and other information, which, however, is rather sparse, the reader is referred to an article on this cable in *The Electrician*, Vol. LIX., p. 217, 1907.

The Anglo-French loaded Four-core telephone cable of 1910 laid by the British Post Office across the English Channel, represents at present (in 1911) the latest achievement in the laying of loaded submarine cables, and the following is a description of this cable taken verbatim from Major O'Meara's valuable paper on the subject :

" The features of the device for loading in the accepted tender are as follows :

" The two double coils required for the four conductors of the cable, each coil being of slightly less than 6 ohms resistance and having an inductance of 0·10 henry at 750 periods per second, are inserted at intervals of 1 knot (1·153 miles), but the two coils nearest the ends of the cable are inserted at a distance of only half a knot from the terminal apparatus, as experiments have shown that in this arrangement reflection losses are considerably reduced. Each double coil consists of two windings on the same iron core, and one winding is connected in series with each conductor (see Fig. 9). By this means the gradual change in permeability in the iron core due to ageing will not affect the balance in the two limbs of the telephone circuit. Each coil is protected with a sheet of metal foil in order to exclude all possibility of the silk covering of the wires of the coils absorbing moisture from the cylindrical envelope of gutta-percha in which they are contained. The cores of the cable are connected to the envelope at its two ends by tapered solid gutta-percha joints. The diameter at the centre of the envelope is 3 inches, and at the cores where the joints terminate 1 inch. An annular rubber distance-piece is inserted between the two coils of a set to give greater flexibility. The total length of the joint is 30·75 inch. As the diameter of the cable at the points where the coils are inserted is increased, a larger number of sheathing wires are required at those points than over the conductors alone. This difficulty is ingeniously overcome by starting a second layer of sheathing wires over the cores, about 27 feet from the centre of the coil envelope, and gradually working them into a single layer with those over the bulge. Finally, they are terminated as a second layer again over the cores at a distance of about 27 feet from the centre of the coil envelope. The method adopted in inserting the coils (British Patent Specification No. 5,547, March, 1907) will perhaps be understood from the diagrams (Fig. 9)."

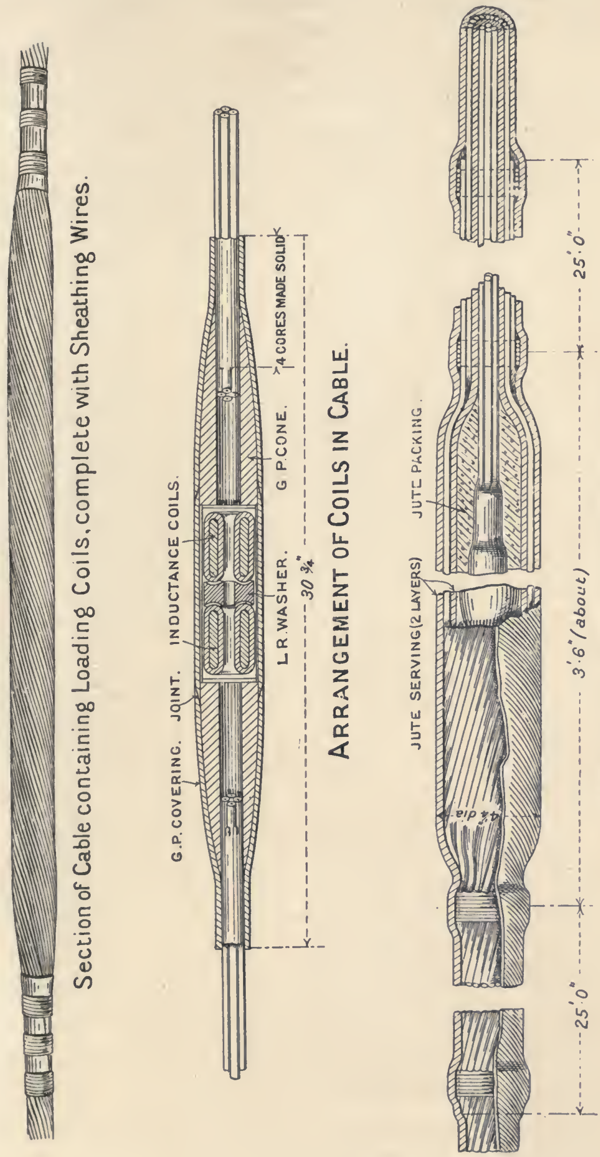

FIG. 9.—Loading Coil inserted in Anglo-French Telephone Cable, 1910.

The following is the Post Office specification for the cable, arrived at after most careful consideration of the problem by the technical experts of the department :

SPECIFICATION FOR ANGLO-FRENCH SUBMARINE TELEPHONE CABLE.

1. *Conductors.*—The conductor of each coil shall be of an approved stranded type, shall weigh not less than 160 lbs. per knot, and shall at a temperature of 75° F. have a resistance not higher than 7·452 standard ohms per knot for a conductor of this gauge. The lay of the stranded conductor shall be left-handed.

2. *Insulator or Dielectric.*—The conductor of each coil shall be insulated by being covered with three alternate layers of Chatterton's compound and gutta-percha, beginning with a layer of the said compound, and no more compound shall be used than may be necessary to secure adhesion between the conductor and the layers of gutta-percha. The dielectric on the conductor of each coil shall weigh not less than 300 lbs. per knot, making the total weight of the conductor of each coil when covered with the dielectric not less than 460 lbs. per knot.

3. *Inductive Capacity.*—The inductive capacity of each coil of such insulated conductor (hereinafter called the core) shall not exceed 0·275 microfarad per knot, and this shall apply equally to the completed cable.

4. *Insertion of Loading Coils.*—The loading coils will be inserted so that diagonal cores in the cable will be used to form a loop or pair, each pair of cores to be fitted with loading coils equally spaced at such distances apart and of such inductance and effective resistance as will make

(*a*) The volume of speech transmitted over a pair of wires in the completed and laid cable at least equal to that through one-seventh of the same length of standard cable, not including terminal losses[1] ;

[1] Standard cable is that having a wire-to-wire capacity for each pair of wires of 0·054 microfarad per statute mile, a loop resistance of 88 ohms per statute mile, and an average insulation resistance of not less than 200 megohms per statute mile wire to wire.

(*b*) The quality of speech or articulation not inferior to that of the speech throughout the standard cable equivalent[1] of the loaded cable pair.

5. *Interference.*—The two loaded cable pairs to be free from telephonic induction or interference, the one from the other, and also from external disturbance from a contiguous cable.

6. *Labelling.*—Each coil of core before being placed in the temperature tank for testing shall be carefully labelled with the exact length of conductor and the exact weight of copper and dielectric respectively which it contains.

7. *Insulation Resistance.*—The insulation resistance of each coil of core, after such coil shall have been kept in water maintained at a temperature of 75° F. for not less than twenty-four consecutive hours immediately preceding the test, shall be not less than 400 nor more than 2,000 megohms per knot when tested at that actual temperature, and after electrification during one minute. The electrification between the first and the second minutes to be not less than 3 nor more than 8 per cent., and to progress steadily. The insulation to be taken not less than fourteen days after manufacture.

Each coil of core may be subjected, before the ordinary insulation test is taken, to an alternating electromotive force of 5,000 volts and 100 complete periods per second for fifteen minutes.

8. *Preservation.*—The core shall during the process of manufacture be carefully protected from sun and heat, and shall not be allowed to remain out of water.

9. *Joints.*—All joints shall be made by experienced workmen, and the contractor shall give timely notice to the Engineer-in-chief or other authorised officer of the Postmaster-General whenever a joint is about to be made, in order that he may test the same. The contractor shall allow time for a thorough testing of each and every joint in the insulated trough by accumulation, and the leakage from any joint during one minute shall be not more than double that from an equal length of the perfect core.

[1] By the standard cable equivalent of any loop is meant the number of statute miles of loop in a standard cable through which the same volume of speech is obtained as through the loop under test.

10. *Taping and Serving.*—The cores to be four in number, and to be stranded with a left-handed lay, and during the process of stranding be wormed with best wet fully tanned jute yarn, so that the whole may be as nearly as possible of a cylindrical form, and shall then be covered (1) with cut cotton tape prepared with ozokerit compound, (2) with pliable brass tape 0·004 inch in thickness and 1 inch in width, and (3) with another serving of cotton tape, similar to the first, the lap in each case being not less than 0·250 inch.

The cores, prepared as above specified, shall then be served with best wet fully tanned jute yarn, sufficient to receive the sheathing, hereafter specified, and no loose threads shall, in the process of sheathing, be run through the closing machine. The cores so served shall be kept in tanned water at ordinary temperature, and shall not be allowed to remain out of water except so far as may be necessary to feed the closing machine.

11. *Sheathing.*—The served core to be sheathed with sixteen galvanised iron wires, each wire having a diameter of 280 mils, or within 3 per cent. thereof above or below the same. The breaking weight of each wire to be not less than 3,500 lbs., with a minimum of ten twists in 6 inches. The length of lay to be 18 inches, and to be left-handed.

The wire to be of homogeneous iron, well and smoothly galvanised with zinc spelter. The galvanising will be tested by taking samples from any coil or coils, and plunging them into a saturated solution of sulphate of copper at 60° F., and allowing them to remain in the solution for one minute, when they will be withdrawn and wiped clean. The galvanising shall admit of this process being four times performed with each sample without there being, as there would be if the coating of zinc were too thin, any sign of a reddish deposit of metallic copper on the wire. If, after the examination of any particular quantity of iron wire, 10 per cent. of such wire does not meet all or any of the foregoing requirements, the whole of such quantity shall be rejected, and no such quantity or any part thereof shall on any account be presented for examination and testing, and this stipulation shall be deemed to be and shall be treated as an essential condition of the contract. Before being used for the sheathing of the cable, the wire shall be heated in a kiln or oven, just sufficiently to drive off all moisture, and whilst warm shall be dipped into pure hot gas-tar (freed from naphtha). The iron wire so dipped shall not be used for sheathing the cable until the coating of gas-tar is thoroughly set. No weld or braze in any one wire of the sheath shall be within six feet of a weld or braze in any other wire. All welds or brazes made during the manufacture of the cable shall be regalvanised and retarred.

12. *Compound and Serving.*—The sheathed cores shall be covered with two coatings of compound and two servings of three-ply jute yarn, the said compound being placed between the two servings and over the outer serving of yarn aforesaid, the two servings of yarn to be laid on in directions contrary to each other.

The compound referred to in this paragraph shall consist of pitch 85 per cent., bitumen $12\frac{1}{2}$ per cent., and resin oil $2\frac{1}{2}$ per cent., and the yarn referred to shall be spun from the best quality of jute, and shall be saturated with gas-tar freed from acid and ammonia, the yarn being thoroughly dried after saturation and before being used, so as to have no superfluous tar adhering.

13. *Measurement and Marks.*—A correct indicator shall be attached to the closing machine, and a mark to be approved by the Engineer-in-chief shall be made on the cable at the termination of each knot of completed cable, and also over each joint or set of joints.

14. *Laying.*—If the tender for laying be accepted, the contractors shall provide the necessary cable-laying ship and all appliances and all apparatus in connection therewith for the laying and testing of the cable during the laying operations. Facilities must be provided for inspection of the work, if considered necessary, by an officer of the Postmaster-General during the progress of the laying operations.

The cable to be laid over the course shown by the dotted red line on the accompanying Admiralty chart, or as hereafter agreed upon.

On completion of the laying operations the spare cable left on board is to be delivered at the Post Office Cable Depôt, Dover, or paid out and buoyed in the sea near Dover, as may be directed by the Engineer-in-chief.

15. The contractors are required to guarantee that the completed cable shall reach and maintain the standard laid down in the specification, and before final acceptance the cable shall be subject to such tests and experiments as the Postmaster-General may deem necessary during the manufacture, laying, and for a period of thirty consecutive days from the completion of the latter.

Major O'Meara states (*loc. cit.*) that "the investigations that had been made left little doubt concerning the balance of advantages in favour of the 'coil' loaded type of cable from the electrical standpoint, but as the expenditure involved was very great, and as it was felt that the main difficulty in connection with this type of cable would be in safely laying the cable at the bottom of the sea, it was considered that special precautions were necessary to ensure that the responsibility for any defects that might be disclosed after it had been laid should be definitely traced to the responsible party. To afford the necessary protection to the department, it seemed desirable to stipulate in the specification that the manufacturers of the cable should also undertake to lay it, and to hand it over *in situ.* This course was approved by the Postmaster-General, and the invitations to tender were issued on these lines. The conditions were accepted by Messrs. Siemens Bros. & Co., who were the successful tenderers.

"It will be recognised that the mechanical problem in connection with this type of cable was more difficult to solve than the electrical problem, as it was necessary that the part of the cable containing the coils should be so designed that it could be paid over the sheaves of the cable-ship without any risk of damage to the coils themselves. However, Major O'Meara said he was glad to say that the manufacturers succeeded in solving this problem in a most satisfactory manner.

"The cable was under the constant supervision of the Post Office Engineering Department during the period of its manufacture, and electrical tests were carried out from time to time. On January 18th, 1910, after the completion of the cable, measurements to determine its attenuation constant were made at the works of Messrs. Siemens Bros. & Co. at Woolwich. The conductors of the cable were joined up so as to provide a metallic circuit of 41·704 knots, and in order to get rid of terminal effects artificial cable was joined to the ends of the loaded cable as shown in Fig. 10.

Current was supplied to this circuit by a generator giving 1·585 volts at a frequency of 750 alternations per second. Readings were taken on a thermo-galvanometer placed successively at A and B, and the attenuation constant was calculated by the formula $I_2 = I_1 \epsilon^{-al}$.

"With ten miles of 'standard' cable (attenuation constant 0·1187 per knot) at each end of the circuit the current values at

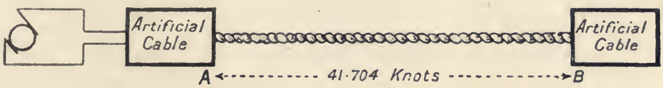

Fig. 10.

A were found to be 0·327 milliampere, and at B 0·172 milliampere, a therefore being 0·0154.

"With fifteen miles of 'standard' cable at each end of the circuit the current values at A were found to be 0·212 milliampere, at B 0·110 milliampere, from which we similarly obtain $a = 0·0152$.

"The volume of the speech transmitted over the loaded cable was also compared with that over an artificial "standard" cable, the electrical constants of which are known. The result of these tests indicated that the attenuation constant of the loaded cable was 0·0147."

The table on p. 288, given by Major O'Meara, supplies the details of the primary constants of this cable both with loading coils inserted and without them, and it also shows the attenuation constants before and after loading.

Mr. W. Dieselhorst was entrusted by Messrs. Siemens Bros. with the actual operation of laying the cable, and Mr. F. Pollard, Submarine Superintendent, Dover, was detailed to watch the interests of the Post Office.

CONSTANTS OF THE ANGLO-FRENCH COIL LOADED TELEPHONE CABLE COMPARED WITH THE CONSTANTS OF THE SAME CABLE WITHOUT LOADING AND CONTINUOUSLY LOADED (REVISED).

Cable Core Details.			Constants per Knot of Loop at 750 p.p.s.					Attenuation Constant.			
Overall Diameter of Copper.	Loading.	Overall Diameter of Gutta-percha.	Resistance. Ohms. Direct Current.	Resistance. Ohms. 750 p.p.s.	Inductance in Millihenrys.	Capacity in Microfarads.	Leakance in Mhos.	Per Kilometre.	Per Statute Mile.	Per Nautical Mile. Calculated.	Per Nautical Mile. Observed.
Mils.		Mils.									
106·2	Three layers of 7·88 mils iron wire, 101·5 turns per inch . .	405[1]	14·95	16·06	22·00	0·181	$2{\cdot}4\times10^{-5}$	0·0124	0·0200	0·0230	—
106·2	Unloaded	390	14·95	14·95	2·00	0·138	$2{\cdot}4\times10^{-5}$	0·0278	0·0447	0·0520	—
106·2	Coils of 100 m.h. every knot . . . . . Steady current resistance, 2·25 ohms . Resistance at 750 p.p.s., 6 ohms . . .	390	17·20	20·90	102·00	0·138	$2{\cdot}4\times10^{-5}$	0·0091	0·0147	0·0166	0·0166

[1] Weight of gutta-percha same as in cases 2 and 3.

FIG. 11.—Passing a Loading Coil of the Anglo-French 1910 Telephone Cable over the Sheave into the Sea.
Laid by Messrs. Siemens Bros. for the British Post Office.

For the full details of the laying of this cable and the manner in which the engineering difficulties were overcome in the manufacture and laying by the contractors, Messrs. Siemens Bros., the reader must consult Major O'Meara's admirable paper on the subject in the *Journal of the Institution of Electrical Engineers.*

The photograph reproduced in Fig. 11 is taken by permission from Major O'Meara's paper (*loc. cit.*), and represents the passing of a loading coil in the 1910 Anglo-French cable over the sheaves of the cable-ship *Faraday* during the process of laying the cable. It will be seen that the type of loading coil adopted does not render the cable to any extent cumbersome and unhandable.

The constants of the cable and some numerical values connected therewith both for the unloaded cable and for the cable with loads are very approximately as follows:

Unloaded Cable per nautical mile.	*Loaded Cable per nautical mile.*
$R=14{\cdot}42$ ohms,	$R=20{\cdot}45$ ohms,
$L=0{\cdot}002$ henry,	$L=0{\cdot}1$ henry,
$C={\cdot}138\times10^{-6}$ farad,	$C={\cdot}138\times10^{-6}$ farad,
$S=2{\cdot}4\times10^{-5}$ mhos.	$S=2{\cdot}4\times10^{-5}$ mhos.

$$n=750,\ p=2\pi n=4{,}710.$$

Hence for the loaded cable we have

$$\sqrt{R^2+p^2L^2}=\sqrt{418+221{,}841}$$

$$\sqrt{S^2+p^2C^2}=10^{-6}\sqrt{576+422{,}500}$$

$$\sqrt{LC}=\sqrt{\frac{138}{10^{10}}},\qquad Lp=471,\qquad Cp=\frac{65}{10^5}.$$

Therefore for the loaded cable

$$\alpha=\sqrt{\frac{RS}{2}}\text{ nearly }=\sqrt{\frac{245}{10^6}}={\cdot}016\text{ (approximately)};$$

$$\beta=p\sqrt{LC}=4{,}710\sqrt{\frac{138}{10^{10}}}={\cdot}542.$$

Hence $$\lambda=\frac{2\pi}{\beta}=11{\cdot}6\text{ nauts},$$

$$\frac{R}{L}=204{\cdot}5\quad\frac{S}{C}=169\quad\sqrt{\frac{S}{C}}=13,$$

and $$\alpha=\sqrt{\frac{RC}{2}}\sqrt{\frac{S}{C}}=13\sqrt{\frac{RC}{2}}.$$

The loading coils, being 1 naut apart, are therefore at the rate of eleven or twelve per wave for the standard wave length, corresponding to a frequency of about 800, and the spacing complies with Pupin's law.

As regards the practical improvement introduced by the loading coils in the above cable the following quotation from Major O'Meara's paper (*loc. cit.*) is interesting and important. He said:

"The cable has been under continuous observation since it was laid, and a large number of tests have been carried out. Particulars of some of them are given in an appendix. It has fortunately been possible to obtain independent testimony on the question of the increase in the range, and in the improvement in the quality of speech transmitted by means of the loaded cable as compared with a similar cable unloaded. Speech tests were made in July last by Messrs. W. R. Cooper, W. Duddell, F.R.S., W. Judd, and J. E. Kingsbury, and the results are interesting. The cable was looped at the French end (Cape Grisnez), and the English ends were connected to two telephone sets, one installed in the cable hut at Abbot's Cliff and the other in the coastguard look-out shelter some 100 feet distant. Graduated artificial cables were provided so that the listener at the cable-hut could insert various values of the 'standard' cable into the circuit until his own limit of satisfactory audibility was reached. It was possible to insert the 'standard' cable values equally at the two ends of the cable (*i.e.*, so as to form a symmetrical circuit in relation to the submarine cable), or unequally, as desired. The results shown in the table below were obtained.

Observer listening.	Old Cable. Added Length of Standard Cable.	New Cable. Added Length of Standard Cable.	Gain by New Cable.
			Miles.
W. R. Cooper	24 miles symmetrical	48 miles symmetrical	24
W. Duddell	24 miles symmetrical	40 miles symmetrical	16
		50 miles symmetrical	26
		55 miles at one end	21
W. Judd	26 miles symmetrical	40 miles symmetrical	14
J. E. Kingsbury	26 miles symmetrical	40 miles symmetrical	14

"The mean gain by the use of the new cable is therefore seventeen miles of 'standard' cable for the standard of audibility accepted as commercial by the four observers named. When the cables were alone in circuit some of the observers noticed that in the case of the new cable there was a distinct improvement in the quality of the speech as compared with the old cable.

"The employment of unloaded 800-lb. copper aerial conductors, such as are in use for the most important long-distance trunk circuits in this country, will render it possible for very satisfactory conversations to take place from call-boxes between centres in England and on the Continent when the added distances from the ends of the cable do not exceed 1,700 miles; that is to say, with land-lines of this description well-maintained conversations between London and Astrakhan on the Caspian Sea would be possible. In his inaugural address to the Institution,[1] Sir John Gavey included a table of equivalents of the various types of unloaded conductors. It may be assumed that in practice aerial conductors of the smaller gauges can be improved by loading twofold, and the conductors in cables threefold, so that it is not difficult to determine the centres between which the new Anglo-French telephone cable will provide communication, assuming that a particular type of conductor is employed to complete the circuit."

5. Effect of Leakance on Loaded Cables.—A brief reference has already been made to the influence of leakance in the case of loaded cables upon the value of the attenuation constant in connection with the doubt thrown upon the possibility of effectively loading gutta-percha insulated cables. This question is important, and must be considered a little more at length. It has been dealt with in a paper by Dr. A. E. Kennelly to which reference has already been made, viz., "On the Distribution of Pressure and Current over Alternating Current Circuits" (see *Harvard Engineering Journal*, 1905—1906), under the heading "Effect of Dielectric Losses on Loading." Dr. Kennelly discusses this matter as follows:

[1] See Sir John Gavey's Inaugural Address, *Journal of the Institution of Electrical Engineers*, Vol. XXXVI., p. 26, 1905.

Let the conductor impedance of the cable, viz., the quantity $R + jpL$, be denoted by $Z_c \underline{/ \theta_c}$ as a vector. Then, equating the sizes, we have

$$Z_c^2 = R^2 + p^2L^2 \text{ and } \tan \theta_c = \frac{pL}{R}.$$

The ratio Lp/R may be called the reactance factor of the conductor at the angular velocity p.

Also the dielectric admittance of the cable, viz., the quantity $S + jpC$, may be denoted as a vector by $Y_D \underline{/ \theta_D}$, and hence

$$Y_D^2 = S^2 + p^2C^2 \text{ and } \tan \theta_D = \frac{Cp}{S}.$$

The ratio of the susceptance Cp to the dielectric conductance S at a particular angular velocity p may be called the susceptance factor of the cable, although cable electricians generally deal more with the quantity $\frac{S}{C}$ as the ratio to be measured. In any case $\frac{Cp}{S}$ is the tangent of the angle of slope of the vector Y_D.

Loading a circuit obviously increases the slope of the vector impedance Z_c. This is particularly noticed in the case of telephone cables, in which when unloaded the reactance factor $\frac{Lp}{R}$ at a frequency of 800 or for $p = 5{,}000$ may be of the order of 0·03 to 0·05, and the vectorial angle θ_c may be $1° \, 30'$ or $2°{\cdot}0$ or so. On the other hand, if there is no dielectric loss S is zero, and the slope of the admittance vector is 90^0, since then its tangent Cp/S is infinite. In such cases we may theoretically diminish the attenuation constant without limit by increasing the inductance of the line per unit of length. For the attenuation constant a is the real part of the product of $\sqrt{R + jpL}$ and $\sqrt{S + jpC}$. The reader should remember that to square-root a vector we have to square-root its size and reduce the slope to half, whilst to obtain the product of two vectors we have to multiply the sizes and add the slopes. Hence, leaving out of account sizes, we may say that if L and S are both very small, then the slope of the conductor impedance vector is nearly zero, and that of the dielectric admittance vector is nearly 90°. Hence the vector representing the square root of their product, or the propagation constant, has a slope of 45°. If we keep S small, but make L very large, then the slope of both impedance and admittance vectors is nearly 90°, and the square root of their product, or the propagation constant, has also a slope of nearly 90°. Hence its horizontal step, or real part which is the attenuation constant, will be small. If, however, S is large, the slope of the admittance vector is much less than 90° and that of its square root much less than 45°, and hence even if the slope of the impedance vector is 90° the slope of the propagation constant is something considerably less than 90°, and that means that the attenuation constant cannot be reduced to zero. In fact, if S is not zero, but has an appreciable value, then it is useless to load the cable beyond the point at which Lp/R becomes equal to Cp/S. For the attenuation constant

$$\alpha=\sqrt{\frac{1}{2}\sqrt{(R^2+p^2L^2)\,(S^2+p^2C^2)}+(RS-p^2LC)},$$

and if we consider R, S, C, and p to be constant and L variable it is very easy to prove in the ordinary way by finding the differential coefficient $\frac{d\alpha}{dL}$ and equating it to zero that the above expression for α has a minimum value when $L=\frac{CR}{S}$, in other words when $\frac{Lp}{R}=\frac{Cp}{S}$, that is when $\theta_c=\theta_D$, or when the cable is distorsionless. If then there is sensible leakance in the dielectric the attenuation constant α cannot be reduced below the value $\alpha=\sqrt{SR}$ which it has when the cable fulfils the Heaviside conditions, $L/R=C/S$, for being distorsionless. It follows then that in the case of loaded cables great care must be taken to keep the leakance S very small, or nearly zero. This accounts for part of the difficulty of loading aerial lines.

If we write down the already-given formula for the attenuation constant α of a cable, viz.,

$$2\alpha^2=\sqrt{(R^2+p^2L^2)\,(S^2+p^2C^2)}+(RS-p^2LC),$$

it is easily transformed into

$$2\alpha^2=\sqrt{S^2R^2\left(1+\frac{p^2L^2}{R^2}\right)\left(1+\frac{p^2C^2}{S^2}\right)}+SR\left(1-\frac{pL}{R}\frac{pC}{S}\right).$$

If then $\frac{pL}{R}=\frac{pC}{S}$, we have $\alpha=\sqrt{SR}$.

If S is absolutely zero, then by making pL or L sufficiently large compared with R we can reduce the value of a indefinitely. But if S has a finite value, then beyond a certain point, viz., when $L = R\frac{C}{S}$, we do not decrease, but actually increase, the value of a.

Accordingly, although in perfectly insulated lines we may with advantage increase almost indefinitely the inductance, provided we do not increase the resistance at the same time; yet in imperfectly insulated lines there is a limit beyond which increase of the inductance increases instead of diminishing the attenuation constant.

The table on p. 296, taken from Dr. Kennelly's paper on "The Distribution of Pressure and Current over Alternating Current Circuits," shows the difference produced in loading a line of absolutely zero leakance up to 200 millihenrys per kilometre and the same loading for a line having an insulation resistance of 10,000 ohms per kilometre, or a leakance of 10^{-4} mhos per kilometre. In the first case the loading produces a remarkable reduction in the attenuation constant, and in the second case it produces very little.

It is abundantly clear, therefore, that a loaded cable must be a well-insulated cable if we are to obtain the benefit of the loading in the form of a small attenuation constant.

It is this fact, combined with the large dielectric current of gutta-percha-covered cable, which threw doubt originally upon the possibility of effectively loading submarine telephone cables insulated with *G.P.* But these doubts have been removed by the success of the 1910 Anglo-French Channel telephone cable.

It is, however, essential to secure good insulation for the loading coils themselves in underground telephone cables. The practice of the National Telephone Company in this matter is to build underground pits at regular intervals of a mile or two, as the case may be, and place in these cast-iron watertight boxes in which are contained the highly insulated loading coils.

The lead-covered paper-insulated cable enclosing many strands or separate pairs of conductors passes through this pit (see

TABLE SHOWING THE EFFECT OF DIELECTRIC LEAKANCE ON ATTENUATION.

Constants of the Cable per Kilometre.				Total Inductance per Kilometre in millihenrys after loading.	Propagation Constant per Kilometre $=\alpha+j\beta$.		Initial Sending End Impedance Z_0.		Attenuation Constant α per Kilometre.
S in mhos.	R in ohms.	L in millihenrys.	Cp		Size.	Slope.	Size.	Slope.	
0	54·57	0·31	4·35	0·31	0·1541	45° 49′	354·3	$\diagdown 44^\circ 11'$	0·1074
0	54·57	0·31	4·35	47·46	0·3254	83° 32′	748·2	$\diagdown 6^\circ 29'$	0·0367
0	54·57	0·31	4·35	60·00	0·3642	84° 51′	837·2	$\diagdown 5^\circ 9'$	0·0327
0	54·57	0·31	4·35	200·00	0·66	88° 27′	1517·	$\diagdown 1^\circ 33'$	0·0179
10^{-4}	54·57	0·31	4·35	0·31	0·1561	39° 30′	349·7	$\diagdown 37^\circ 43'$	0·1208
10^{-4}	54·57	0·31	4·35	47·46	0·3297	77° 03′	738·7	0°	0·0742
10^{-4}	54·57	0·31	4·35	60·00	0·3689	78° 22′	826·6	$\diagup 1^\circ 19'$	0·0744
10^{-4}	54·57	0·31	4·35	200·00	0·6686	81° 58′	1471·	$\diagup 4^\circ 55'$	0·0935

Fig. 8), and the coils are connected into the different circuits. In this manner good insulation is secured for the line and coils.

The attenuation constant of the loaded line can always be calculated very approximately by the formula

$$a=\frac{1}{2}\left(R+\frac{SL}{C}\right)\sqrt{\frac{C}{L}}.$$

This formula is arrived at in the following manner:

By the binomial theorem we have for the expansion of a binomial $(a+n)^n$ the series

$$(a+x)^n=a^n+na^{n-1}x+\frac{n\,.\,n-1}{1\,.\,2}a^{n-2}n^2+\text{etc.}$$

If $n=\frac{1}{2}$, then

$$\sqrt{a+x}=(a+x)^{\frac{1}{2}}=\sqrt{a}+\frac{x}{2\sqrt{a}}-\frac{x^2}{8a^{\frac{3}{2}}}+\text{etc.}$$

Hence if x is small compared with a, so that we can neglect powers of x/a, we have $\sqrt{a+x}=\sqrt{a}+\frac{x}{2\sqrt{a}}$ nearly.

Accordingly, if R is small compared with pL and S is small compared with pC, we have

$$\sqrt{R^2+p^2L^2}=pL+\frac{R}{2}\frac{R}{pL}$$

and

$$\sqrt{S^2+p^2C^2}=pC+\frac{S}{2}\frac{S}{pC}.$$

Since, then, $2\alpha^2=\sqrt{R^2+p^2L^2}\sqrt{S^2+p^2C}+SR-p^2LC$, it follows that when R/pL and S/pC are both small quantities compared with unity we have

$$2\alpha^2=\frac{S^2}{2}\frac{L}{C}+\frac{R^2}{2}\frac{C}{L}+SR,$$

or

$$\alpha=\frac{1}{2}\left(R+L\frac{S}{C}\right)\sqrt{\frac{C}{L}}.$$

Accordingly the attenuation is greatly affected by the value of S/C.

No really satisfactory method has yet been found for measuring the value of the leakance S or the ratio S/C for telephonic frequencies, but it is found that by taking $S/C=80$ this formula gives attenuation constants which are in close agreement with observed values for loaded cables. Thus, in a discussion on a paper by Professor Perry on "Telephone Circuits," Mr. A. W. Martin, of the General Post Office, gave some useful measurements confirming this result for loaded cables.

Cables of various lengths were loaded with iron-cored inductance coils, each having effective resistances of 5·4 ohms at 750 frequency and 15·0 ohms at 2,000 and 3·5 ohms for steady currents, also an inductance of 0·135 henry per coil. These coils were inserted at various intervals in a line of conductor resistance 18 ohms per mile of loop, and capacity 0·055 m.f.d., and inductance 0·001 henry per mile of loop. The attenuation constants were then calculated from the above formula, taking $S/C = 80$, and they were also measured, and the results were as follows:

Interval between Loading Coils in miles.	Attenuation Constants for Frequency 750.		Coils per Wave at a Frequency of 2,000.	Articulation.
	Calculated.	Observed.		
1·1	0·011	0·013	5·6	Very good
2·1	0·012	0·012	4·0	Very good
3·2	0·013	0·012	3·3	Good
4·3	0·014	0·014	2·8	Bad
Unloaded	0·042	0·045	—	—

In the case of the Anglo-French telephone cable (1910) above described, the observed attenuation constant corresponds to a value of $S/C = 99$ instead of 80. There is no doubt that the ratio of S/C for any telephone conductor plays a very important part in determining the speech-transmitting efficiency.

In the United States one of the principal difficulties in connection with the loading of long distance aerial telephone lines has been the leakage over the insulators, and a more efficient type of glass insulator has had to be substituted for the ordinary type in order to keep down the leakage, which prevents the loading from having its full effect.

The reader will find a considerable amount of valuable information on the properties of loaded lines in the discussion which took place at the Physical Society of London on a paper by Professor Perry in 1910 (see *The Electrician*, March 11th, 1910, p. 879), and also a longer and even more important discussion which took place at the Institution of Electrical Engineers on the paper by Major O'Meara on "Submarine Cables for Long Distance Telephone Circuits" (see *The Electrician*, Vol. LXV., p. 609, 1910, and Vol. LXVI., pp. 375, 417, 419, 589, and 615, 1911), in which all the leading experts in telephony and telegraphy in England took part.

APPENDIX.

The table below is taken by kind permission from a paper by Dr. A. E. Kennelly, published in the *Harvard Engineering Journal*, May, 1903.

TABLE OF SINES, COSINES, TANGENTS, COTANGENTS, SECANTS AND COSECANTS OF HYPERBOLIC ANGLES.

The Sines, Cosines, and Tangents have been taken from Ligowski's Tables published in Berlin in 1890. The Cotangents, Secants, and Cosecants have been deduced from the preceding quantities.

u.	Sinh. u.	Cosh. u.	Tanh. u.	Coth. u.	Sech. u.	Cosech. u.	u.
0·00	0·	1·000	0·	∞	1·00	∞	**0·00**
0·01	0·010000	1·000050	0·01000	100·	0·9999	100·	0·01
0·02	0·020001	1·000200	0·02000	50·	0·9998	50·	0·02
0·03	0·030005	1·000450	0·02999	33·34	0·9995	33·333	0·03
0·04	0·040011	1·000800	0·03998	25·013	0·9992	24·99	0·04
0·05	0·050021	1·001250	0·04996	20·016	0·9987	19·992	0·05
0·06	0·060036	1·001801	0·05993	16·686	0·9982	16·657	0·06
0·07	0·070057	1·002451	0·06989	14·308	0·9975	14·274	0·07
0·08	0·080085	1·003202	0·07983	12·527	0·9968	12·487	0·08
0·09	0·090122	1·004053	0·08976	11·141	0·9959	11·097	0·09
0·10	0·100167	1·005004	0·09967	10·033	0·9950	9·983	**0·10**
0·11	0·110222	1·006056	0·10956	9·128	0·9940	9·073	0·11
0·12	0·120288	1·007209	0·11943	8·373	0·9928	8·314	0·12
0·13	0·130366	1·008462	0·12927	7·735	0·9916	7·669	0·13
0·14	0·140458	1·009816	0·13909	7·189	0·9902	7·120	0·14
0·15	0·150563	1·011271	0·14888	6·716	0·9888	6·642	0·15
0·16	0·160684	1·012827	0·15865	6·303	0·9873	6·223	0·16
0·17	0·170820	1·014485	0·16838	5·939	0·9857	5·854	0·17
0·18	0·180974	1·016244	0·17808	5·615	0·9840	5·525	0·18
0·19	0·191145	1·018104	0·18775	5·325	0·9822	5·232	0·19

Table of Sines, Cosines, Tangents, Cotangents, Secants and Cosecants of Hyperbolic Angles.—*continued.*

u.	Sinh. u.	Cosh. u.	Tanh. u.	Coth. u.	Sech. u.	Cosech. u.	u.
0·20	0·201336	1·020067	0·19737	5·067	0·9803	4·967	**0·20**
0·21	0·211547	1·022131	0·20696	4·832	0·9784	4·726	0·21
0·22	0·221779	1·024298	0·21652	4·618	0·9763	4·509	0·22
0·23	0·232033	1·026567	0·22603	4·425	0·9742	4·310	0·23
0·24	0·242311	1·028939	0·23549	4·246	0·9719	4·127	0·24
0·25	0·252612	1·031413	0·24492	4·083	0·9695	3·959	0·25
0·26	0·262939	1·033991	0·25430	3·932	0·9671	3·803	0·26
0·27	0·273292	1·036672	0·26363	3·793	0·9646	3·659	0·27
0·28	0·283673	1·039457	0·27290	3·664	0·9620	3·525	0·28
0·29	0·294082	1·042346	0·28214	3·544	0·9591	3·400	0·29
0·30	0·304520	1·045339	0·29131	3·433	0·9566	3·284	**0·30**
0·31	0·314989	1·048436	0·30043	3·328	0·9537	3·175	0·31
0·32	0·325489	1·051638	0·30951	3·231	0·9511	3·072	0·32
0·33	0·336022	1·054946	0·31852	3·140	0·9479	2·976	0·33
0·34	0·346589	1·058359	0·32748	3·053	0·9447	2·885	0·34
0·35	0·357190	1·061878	0·33637	2·973	0·9416	2·800	0·35
0·36	0·367827	1·065503	0·34522	2·897	0·9385	2·719	0·36
0·37	0·378500	1·069234	0·35399	2·825	0·9353	2·642	0·37
0·38	0·389212	1·073073	0·36271	2·757	0·9319	2·569	0·38
0·39	0·399962	1·077019	0·37136	2·693	0·9285	2·500	0·39
0·40	0·410752	1·081072	0·37995	2·632	0·9250	2·434	**0·40**
0·41	0·421584	1·085234	0·38847	2·574	0·9215	2·372	0·41
0·42	0·432457	1·089504	0·39693	2·512	0·9178	2·312	0·42
0·43	0·443374	1·093883	0·40532	2·467	0·9141	2·256	0·43
0·44	0·454335	1·098372	0·41365	2·417	0·9103	2·201	0·44
0·45	0·465342	1·102970	0·42190	2·370	0·9066	2·149	0·45
0·46	0·476395	1·107679	0·43009	2·325	0·9025	2·099	0·46
0·47	0·487496	1·112498	0·43820	2·282	0·8988	2·051	0·47
0·48	0·498646	1·117429	0·44624	2·241	0·8949	2·006	0·48
0·49	0·509845	1·122471	0·45421	2·202	0·8909	1·961	0·49
0·50	0·521095	1·127626	0·46211	2·164	0·8868	1·919	**0·50**
0·51	0·532398	1·132893	0·46995	2·128	0·8827	1·878	0·51
0·52	0·543754	1·138274	0·47769	2·093	0·8785	1·839	0·52
0·53	0·555164	1·143769	0·48538	2·060	0·8743	1·801	0·53
0·54	0·566629	1·149378	0·49299	2·028	0·8700	1·765	0·54
0·55	0·578152	1·155101	0·50052	1·998	0·8658	1·730	0·55
0·56	0·589732	1·160941	0·50797	1·969	0·8614	1·696	0·56
0·57	0·601371	1·166896	0·51536	1·940	0·8570	1·663	0·57
0·58	0·613070	1·172968	0·52266	1·913	0·8525	1·631	0·58
0·59	0·624831	1·179158	0·52990	1·887	0·8480	1·601	0·59

TABLE OF SINES, COSINES, TANGENTS, COTANGENTS, SECANTS AND COSECANTS OF HYPERBOLIC ANGLES.—*continued.*

u.	Sinh. *u.*	Cosh. *u.*	Tanh. *u.*	Coth. *u.*	Sech. *u.*	Cosech. *u.*	*u.*
0·60	0·636654	1·185465	0·53704	1·862	0·8435	1·571	**0·60**
0·61	0·648540	1·191891	0·54413	1·838	0·8390	1·542	0·61
0·62	0·660492	1·198436	0·55112	1·814	0·8344	1·514	0·62
0·63	0·672509	1·205101	9·55805	1·792	0·8298	1·487	0·63
0·64	0·684594	1·211887	0·56490	1·770	0·8251	1·461	0·64
0·65	0·696748	1·218793	0·57166	1·749	0·8205	1·435	0·65
6·66	0·708970	1·225882	0·57836	1·729	0·8158	1·410	0·66
0·67	0·721264	2·232973	0·58498	1·709	0·8110	1·387	0·67
0·68	0·733630	1·240247	0·59152	1·690	0·8065	1·363	0·68
0·69	0·746070	1·247646	0·59798	1·672	0·8015	1·340	0·69
0·70	0·758584	1·255169	0·60437	1·655	0·7967	1·318	**0·70**
0·71	0·771174	1·262818	0·61067	1·637	0·7919	1·297	0·71
0·72	0·783840	1·270593	0·61691	1·621	0·7870	1·276	0·72
0·73	0·796586	1·278495	0·62306	1·605	0·7821	1·255	0·73
0·74	0·809411	1·286525	0·62914	1·590	0·7773	1·235	0·74
0·75	0·822317	1·294683	0·63516	1·574	0·7724	1·216	0·75
0·76	0·835305	1·302971	0·64108	1·5599	0·7675	1·1972	0·76
0·77	0·848377	1·311390	0·64693	1·5457	0·7625	1·1787	0·77
0·78	0·861533	1·319939	0·65271	1·5320	0·7576	1·1607	0·78
0·79	0·874776	1·328621	0·65842	1·5188	0·7527	1·1431	0·79
0·80	0·888106	1·337435	0·66403	1·5059	0·7477	1·1259	**0·80**
0·81	0·901525	1·346383	0·66959	1·4934	0·7427	1·1092	0·81
0·82	0·915034	1·355466	0·67507	1·4813	0·7377	1·0928	0·82
0·83	9·928635	1·364684	0·68047	1·4696	0·7327	1·0768	0·83
0·84	0·942328	0·374039	0·68580	1·4582	0·7278	1·0612	0·84
0·85	0·956116	1·383531	0·69107	1·4470	0·7228	1·0459	0·85
0·86	0·969999	1·393161	0·69626	1·4362	0·7178	1·0309	0·86
0·87	0·983980	1·402931	0·70137	1·4258	0·7128	1·0163	0·87
0·88	0·998058	1·412841	0·70642	1·4156	0·7078	1·0020	0·88
0·89	1·012237	1·422893	0·71139	1·4057	0·7028	0·9881	0·89
0·90	1·026517	1·433086	0·71629	1·3961	0·6978	0·9737	**0·90**
0·91	1·040899	4·443423	0·72114	1·3867	0·6928	0·9607	0·91
0·92	1·055386	1·453905	0·72591	1·3776	0·6878	0·9475	0·92
0·93	1·069978	1·464531	0·73060	1·3687	0·6828	0·9346	0·93
0·94	1·084677	1·475305	0·73522	1·3600	0·6778	0·9219	0·94
0·95	1·099484	1·486225	0·73979	1·3517	0·6728	0·9095	0·95
0·96	1·114402	1·497295	0·74427	1·3436	0·6678	0·8973	0·96
0·97	1·129431	1·508514	0·74870	1·3356	0·6629	0·8854	0·97
0·98	1·144573	1·519884	0·75306	1·3279	0·6579	0·8737	0·98
0·99	1·159829	1·531406	0·75736	1·3204	0·6529	0·8621	0·99

TABLE OF SINES, COSINES, TANGENTS, COTANGENTS, SECANTS AND COSECANTS OF HYPERBOLIC ANGLES.—*continued.*

u.	Sinh. u.	Cosh. u.	Tanh. u.	Coth. u.	Sech. u.	Cosech. u.	u.
1·00	1·175201	1·543081	0·76159	1·3130	0·6480	0·8509	**1·00**
1·01	1·190691	1·554910	0·76576	1·3059	0·6431	0·8395	1·01
1·02	1·206300	1·566895	0·76987	1·2989	0·6382	0·8290	1·02
1·03	1·222029	1·579036	0·77391	1·2921	0·6333	0·8183	1·03
1·04	1·237881	1·591336	0·77789	1·2855	0·6284	0·8078	1·04
1·05	1·253857	1·603794	0·78181	1·2791	0·6235	0·7975	1·05
1·06	1·269958	1·616413	0·78566	1·2728	0·6186	0·7874	1·06
1·07	1·286185	1·629194	0·78846	1·2666	0·6138	0·7777	1·07
1·08	1·302542	1·642138	0·79320	1·2607	0·6090	0·7677	1·08
1·09	1·319029	1·655245	0·79688	1·2549	0·6042	0·7581	1·09
1·10	1·335647	1·668519	0·80050	1·2492	0·5993	0·7487	**1·10**
1·11	1·352400	1·681959	0·80406	1·2437	0·5945	0·7393	1·11
1·12	1·369287	1·695567	0·80757	1·2382	0·5898	0·7302	1·12
1·13	1·386312	1·709345	0·81102	1·2330	0·5850	0·7215	1·13
1·14	1·403475	1·723294	0·81441	1·2279	0·5803	0·7125	1·14
1·15	1·420778	1·737415	0·81775	1·2229	0·5755	0·7038	1·15
1·16	1·438224	1·751710	0·82104	1·2180	0·5708	0·6953	1·16
1·17	1·455813	1·766180	0·82427	1·2132	0·5662	0·6869	1·17
1·18	1·473548	1·780826	0·82745	1·2085	0·5616	0·6786	1·18
1·19	1·491430	1·795651	0·83058	1·2040	0·5569	0·6705	1·19
1·20	1·509461	1·810656	0·83365	1·1995	0·5523	0·6625	**1·20**
1·21	1·527644	1·825841	0·83668	1·1952	0·5477	0·6546	1·21
1·22	1·545979	1·841209	0·83965	1·1910	0·5431	0·6468	1·22
1·23	1·564468	1·856761	0·84258	1·1868	0·5385	0·6392	1·23
1·24	1·583115	1·872499	0·84546	1·1828	0·5340	0·6317	1·24
1·25	1·601919	1·888424	0·84828	1·1789	0·5296	0·6242	1·25
1·26	1·620884	1·904538	0·85106	1·1750	0·5251	0·6170	1·26
1·27	1·640010	1·920842	0·85380	1·1712	0·5206	0·6098	1·27
1·28	1·659301	1·937339	0·85648	1·1675	0·5162	0·6026	1·28
1·29	1·678758	1·954029	0·85913	1·1640	0·5118	0·5957	1·29
1·30	1·698382	1·970914	0·86172	1·1604	0·5074	0·5888	**1·30**
1·31	1·718177	1·987997	0·86428	1·1570	0·5030	0·5820	1·31
1·32	1·738143	2·005278	0·86678	1·1537	0·4987	0·5753	1·32
1·33	1·758283	2·022760	0·86925	1·1504	0·4944	0·5687	1·33
1·34	1·778599	2·040445	0·87167	1·1472	0·4901	0·5623	1·34
1·35	1·799093	2·058333	0·87405	1·1441	0·4858	0·5559	1·35
1·36	1·819766	2·076427	0·87639	1·1410	0·4816	0·5495	1·36
1·37	1·840622	2·094729	0·87869	1·1380	0·4773	0·5433	1·37
1·38	4·861662	2·113240	0·88095	1·1351	0·4732	0·5372	1·38
1·39	1·882887	2·131963	0·88317	1·1323	0·4690	0·5311	1·39

TABLE OF SINES, COSINES, TANGENTS, COTANGENTS, SECANTS AND COSECANTS OF HYPERBOLIC ANGLES.—*continued.*

u.	Sinh. u.	Cosh. u.	Tanh. u.	Coth. u.	Sech. u.	Cosech. u.	u.
1·40	1·904302	2·150898	0·88535	1·1295	0·4649	0·5252	**1·40**
1·41	1·925906	2·170049	0·88749	1·1268	0·4608	0·5192	1·41
1·42	1·947703	2·189417	0·88960	1·1241	0·4568	0·5134	1·42
1·43	1·969695	2·209004	0·89167	1·1215	0·4527	0·5077	1·43
1·44	1·991884	3·228812	0·89370	1·1189	0·4486	0·5020	1·44
1·45	2·014272	2·248842	0·89569	1·1165	0·4446	0·4964	1·45
1·46	2·036862	2·269098	0·89765	1·1140	0·4407	0·4909	1·46
1·47	2·059655	2·289580	0·89958	1·1116	0·4367	0·4855	1·47
1·48	2·082654	2·310292	0·90147	1·1093	0·4329	0·4802	1·48
1·49	2·105861	2·331234	0·90332	1·1070	0·4290	0·4749	1·49
1·50	2·129279	2·352410	0·90515	1·1048	0·4251	0·4697	**1·50**
1·51	2·152910	1·373820	0·90694	1·1026	0·4212	0·4645	1·51
1·52	2·176757	2·395469	0·90870	1·1005	0·4174	0·4594	1·52
1·53	2·200821	2·417356	0·91042	1·0984	0·4137	0·4543	1·53
1·54	2·225105	2·439486	0·91212	1·0963	0·4099	0·4494	1·54
1·55	2·249611	2·461859	0·91379	1·0943	0·4062	0·4444	1·55
1·56	2·274343	2·484479	0·91542	1·0924	0·4025	0·4398	1·56
1·57	2·299302	2·507347	0·91703	1·0905	0·3988	0·4350	1·57
1·58	2·324490	2·530465	0·91860	1·0886	0·3952	0·4302	1·58
1·59	2·349912	2·553837	0·92015	1·0868	0·3916	0·4255	1·59
1·60	2·375568	2·577464	0·92167	1·0850	0·3879	0·4209	**1·60**
1·61	2·401462	2·601349	0·92316	1·0832	0·3844	0·4164	1·61
1·62	2·427596	2·625495	0·92462	1·0815	0·3809	0·4119	1·62
1·63	2·453973	2·649902	0·92606	1·0798	0·3774	0·4075	1·63
1·64	2·480595	2·674575	0·92747	1·0782	0·3739	0·4031	1·64
1·65	2·507465	2·699515	0·92886	1·0765	0·3704	0·3988	1·65
1·66	2·534586	2·724725	0·93022	1·0750	0·3670	0·3945	1·66
1·67	2·561960	2·750207	0·93155	1·0735	3·3636	0·3903	1·67
1·68	2·589591	2·775965	0·93286	1·0719	0·3602	0·3862	1·68
1·69	2·617481	2·802000	0·93415	1·0704	0·3569	0·3820	1·69
1·70	2·645632	2·828315	0·93541	1·0690	0·3536	0·3780	**1·70**
1·71	2·674048	2·854914	0·93665	1·0676	0·3503	0·3740	1·71
1·72	2·702731	2·891797	0·93786	1·0662	0·3470	0·3700	1·72
1·73	2·731685	2·908969	0·93906	1·0649	0·3438	0·3661	1·73
1·74	2·760912	2·936432	0·94023	1·0636	0·3405	0·3622	1·74
1·75	2·790414	2·964188	0·94138	1·0623	0·3373	0·3584	1·75
1·76	2·820196	2·992241	0·94250	1·0610	0·3342	0·3546	1·76
1·77	2·850260	3·020593	0·94361	1·0597	0·3310	0·3508	1·77
1·78	2·880609	3·049247	0·94470	1·0585	0·3279	0·3471	1·78
1·79	2·911246	3·078206	0·94576	1·0573	0·3248	0·3435	1·79

Table of Sines, Cosines, Tangents, Cotangents, Secants and Cosecants of Hyperbolic Angles.—*continued.*

u.	Sinh. *u.*	Cosh. *u.*	Tanh. *u.*	Coth. *u.*	Sech. *u.*	Cosech. *u.*	*u.*
1·80	2·942174	3·107473	0·94681	1·0561	0·3218	0·3399	**1·80**
1·81	2·973397	3·137051	0·94783	1·0550	0·3187	0·3363	1·81
1·82	3·004916	3·166942	0·94884	1·0539	0·3158	0·3328	1·82
1·83	3·036737	3·197150	0·94983	1·0528	0·3128	0·3293	1·83
1·84	3·068860	3·227678	0·95080	1·0517	0·3098	0·3258	1·84
1·85	3·101291	3·258528	0·95175	1·0507	0·3069	0·3224	1·85
1·86	3·134032	3·289705	0·95268	1·0497	0·3040	0·3191	1·86
1·87	3·167086	3·321210	0·95359	1·0487	0·3011	0·3157	1·87
1·88	3·200457	3·353047	0·95449	1·0477	0·2982	0·3125	1·88
1·89	3·234148	3·385220	0·95537	1·0467	0·2954	0·3092	1·89
1·90	3·268163	3·417732	0·95624	1·0457	0·2926	0·3059	**1·90**
1·91	3·302504	3·450585	0·95709	1·0448	0·2897	0·3028	1·91
1·92	3·337176	3·483783	0·95792	1·0439	0·2870	0·2997	1·92
1·93	3·372181	3·517329	0·95873	1·0430	0·2843	0·2965	1·93
1·94	3·407524	3·551227	0·95953	1·0422	0·2816	0·2935	1·94
1·95	3·443207	3·585481	0·96032	1·0413	0·2789	0·2904	1·95
1·96	3·479234	3·620093	0·96109	1·0405	0·2762	0·2874	1·96
1·97	3·515610	3·655067	0·96185	1·0397	0·2736	0·2844	1·97
1·98	3·552337	3·690406	0·96259	1·0389	0·2710	0·2815	1·98
1·99	3·589419	3·726115	0·96331	1·0380	0·2684	0·2786	1·99
2·00	3·626860	3·762196	0·96403	1·0373	0·2658	0·2757	**2·00**
2·01	3·66466	3·79865	0·96473	1·0365	0·2632	0·2729	2·01
2·02	3·70283	3·83549	0·96541	1·0358	0·2607	0·2701	2·02
2·03	3·74138	3·87271	0·96608	1·0351	0·2582	0·2673	2·03
2·04	3·78029	3·91032	0·96675	1·0344	0·2557	0·2645	2·04
2·05	3·81958	3·94832	0·96740	1·0337	0·2533	0·2618	2·05
2·06	3·85926	3·98671	0·96803	1·0330	0·2508	0·2596	2·06
2·07	3·89932	4·02550	0·96865	1·0323	0·2484	0·2565	2·07
2·08	3·93977	4·06470	0·96926	1·0317	0·2460	0·2538	2·08
2·09	3·98061	4·10430	0·96986	1·0310	0·2436	0·2512	2·09
2·10	4·02186	4·14431	0·97045	1·0304	0·2413	0·2486	**2·10**
2·11	4·06350	4·18474	0·97101	1·0298	0·2389	0·2461	2·11
2·12	4·10555	4·22558	0·97159	1·0293	0·2366	0·2436	2·12
2·13	4·14801	4·26685	0·97215	1·0286	0·2344	0·2411	2·13
2·14	4·19089	4·30855	0·97274	1·0280	0·2321	0·2386	2·14
2·15	4·23419	4·35067	0·97323	1·0275	0·2298	0·2362	2·15
2·16	4·27791	4·39323	0·97375	1·0269	0·2276	0·2338	2·16
2·17	4·32205	4·43623	0·97426	1·0264	0·2254	0·2314	2·17
2·18	4·36663	4·47967	0·97477	1·0259	0·2232	0·2290	2·18
2·19	4·41165	4·52356	0·97524	1·0254	0·2211	0·2267	2·19

TABLE OF SINES, COSINES, TANGENTS, COTANGENTS, SECANTS AND COSECANTS OF HYPERBOLIC ANGLES.—*continued.*

u.	Sinh. u.	Cosh. u.	Tanh. u.	Coth. u.	Sech. u.	Cosech. u.	u.
2·20	4·45711	4·56791	0·97574	1·0249	0·2189	0·2244	**2·20**
2·21	4·50301	4·61271	0·97622	1·0243	0·2168	0·2221	2·21
2·22	4·54936	4·65797	0·97668	1·0239	0·2147	0·2198	2·22
2·23	4·59617	4·70370	0·97714	1·0234	0·2126	0·2176	2·23
2·24	4·64344	4·74989	0·97758	1·0229	0·2105	0·2154	2·24
2·25	4·69117	4·79657	0·97803	1·0224	0·2085	0·2132	2·25
2·26	4·73937	4·84372	0·97847	1·0220	0·2064	0·2110	2·26
2·27	4·78804	4·89136	0·97888	1·0216	0·2044	0·2089	2·27
2·28	4·83720	4·93948	0·97929	1·0211	0·2024	0·2067	2·28
2·29	4·88683	4·98810	0·97970	1·0207	0·2005	0·2047	2·29
2·30	4·93696	5·03722	0·98010	1·0203	0·1985	0·2026	**2·30**
2·31	4·98758	5·08684	0·98049	1·0199	0·1966	0·2005	2·31
2·32	5·03870	5·13697	0·98087	1·0195	0·1947	0·1985	2·32
2·33	5·09032	5·18762	0·98124	1·0191	0·1928	0·1965	2·33
2·34	5·14245	5·23879	0·98161	1·0187	0·1909	0·1945	2·34
2·35	5·19510	5·29047	0·98198	1·0183	0·1890	0·1925	2·35
2·36	5·24827	5·34269	0·98233	1·0180	0·1872	0·1905	2·36
2·37	5·30196	5·39544	0·98268	1·0177	0·1854	0·1886	2·37
2·38	5·35618	5·44873	0·98302	1·0173	0·1835	0·1867	2·38
2·39	5·41093	5·50256	0·98335	1·0169	0·1817	0·1848	2·39
2·40	5·46623	5·55695	0·98368	1·0166	0·1800	0·1829	**2·40**
2·41	5·52207	5·61189	0·98399	1·0163	0·1782	0·1811	2·41
2·42	5·57847	5·66739	0·98431	1·0159	0·1765	0·1793	2·42
2·43	5·63542	5·72346	0·98462	1·0156	0·1747	0·1775	2·43
2·44	5·69294	5·78010	0·98492	1·0153	0·1730	0·1757	2·44
2·45	5·75103	5·83732	0·98522	1·0150	0·1713	0·1739	2·45
2·46	5·80969	5·89512	0·98551	1·0147	0·1696	0·1721	2·46
2·47	5·86893	5·95352	0·98579	1·0144	0·1680	0·1704	2·47
2·48	5·92876	6·01250	0·98607	1·0141	0·1663	0·1687	2·48
2·49	5·98918	6·07209	0·98635	1·0138	0·1647	0·1670	2·49
2·50	6·05020	6·13229	0·98661	1·0135	0·1631	0·1653	**2·50**
2·6	6·69473	6·76901	0·98403	1·0110	0·1477	0·1494	**2·6**
2·7	7·40626	7·47347	0·99101	1·0091	0·1338	0·1350	**2·7**
2·8	8·19192	8·25273	0·99263	1·0074	0·1212	0·1221	**2·8**
2·9	9·05956	9·11458	0·99396	1·0060	0·1097	0·1104	**2·9**
3·0	10·01787	10·06766	0·99505	1·0050	0·0937	0·09982	**3·0**

Table of Sines, Cosines, Tangents, Cotangents, Secants and Cosecants of Hyperbolic Angles.—*continued.*

u.	Sinh. u.	Cosh. u.	Tanh. u.	Coth. u.	Sech. u.	Cosech. u.	u.
3·1	11·07645	11·12150	0·99595	1·0041	0·0899	0·0903	**3·1**
3·2	12·24588	12·28665	0·99668	1·0033	0·0814	0·0816	**3 2**
3·3	13·53788	13·57476	0·99728	1·0027	0·0736	0·0739	**3·3**
3·4	14·96536	14·99874	0·99778	1·0022	0·0667	0·0668	**3·4**
3·5	16·54263	16·57282	0·99818	1·0018	0·0604	0·0604	**3·5**
3·6	18·28546	18·31278	0·99851	1·0015	0·0646	0·0547	**3·6**
3·7	20·21129	20·23601	0·99878	1·0012	0·0494	0·0495	**3·7**
3·8	22·33941	22·36178	0·99900	1·0010	0·0447	0·0448	**3·8**
3·9	24·69110	24·71135	0·99918	1·0008	0·0405	0·0405	**3·9**
4·0	27·28992	27·30823	0·99933	1·0007	0·0366	0·0366	**4·0**
4·1	30·16186	30·17843	0·99945	1·0006	0·0331	0·0332	**4·1**
4·2	33·33567	33·35066	0·99955	1·0005	0·0300	0·0300	**4·2**
4·3	36·84311	36·85668	0·99963	1·0004	0·0271	0·0271	**4·3**
4·4	40·71930	40·73157	0·99970	1·0003	0·0245	0·0245	**4·4**
4·5	45·00301	45·01412	0·99975	1·0003	0·0222	0·0222	**4·5**
4·6	49·73713	49·74718	0·99980	1·0002	0·0201	0·0201	**4·6**
4·7	54·96904	54·97813	0·99983	1·0002	0·0182	0·0182	**4·7**
4·8	60·75109	60·75932	0·99986	1·0001	0·0165	0·0165	**4·8**
4·9	67·14117	67·14861	0·99989	1·0001	0·0149	0·0149	**4·9**
5·0	74·20321	74·20995	0·99991	1·0001	0·0135	0·0135	**5·0**
5·1	82·0079	82·0140	0·99993	1·00007	0·01219	0·01219	**5·1**
5·2	90·6334	90·6389	0·99993	1·00007	0·01103	0·01103	**5·2**
5·3	100·1659	100·1709	0·99994	1·00006	0·00998	0·00998	**5·3**
5·4	110·7009	110·7055	0·99995	1·00005	0·00903	0·00903	**5·4**
5·5	122·3439	122·3480	0·99996	1·00004	0·00818	0·00818	**5·5**
5·6	135·2114	135·2150	0·99997	1·00003	0·00740	0·00740	**5·6**
5·7	149·4320	149·4354	0·99998	1·00002	0·00669	0·00669	**5·7**
5·8	165·1483	165·1513	0·99998	1·00002	0·00606	0·00606	**5·8**
5·9	182·5174	182·5201	0·99998	1·00002	0·00548	0·00548	**5·9**
6·0	201·7132	201·7156	0·99999	1·00001	0·00496	0·00496	**6·0**
6·1	222·9278	222·9300	1·	1·	0·00449	0·00449	**6·1**
6·2	246·3735	246·3755	1·	1·	0·00406	0·00406	**6·2**
6·3	272·2850	272·2869	1·	1·	0·00367	0·00367	**6·3**
6·4	300·9217	300·9233	1·	1·	0 00332	0·00332	**6·4**
6·5	332·5701	332·5716	1·	1·	0·00301	0·00301	**6·5**
6·6	367·5469	367·5483	1·	1·	0·00272	0·00272	**6·6**
6·7	406·2023	406·2035	1·	1·	0·00246	0·00246	**6·7**
6·8	448·9231	448·9242	1·	1·	0·00223	0·00223	**6·8**
6·9	496·1369	496·1879	1·	1·	0·00202	0·00202	**6·9**

TABLE OF SINES, COSINES, TANGENTS, COTANGENTS, SECANTS AND COSECANTS OF HYPERBOLIC ANGLES.—*continued.*

u.	Sinh. *u.*	Cosh. *u.*	Tanh. *u.*	Coth. *u.*	Sech. *u.*	Cosech. *u.*	*u.*
7·0	548·3161	548·3170	1·	1·	0·00182	0·00182	**7·0**
7·1	605·9831	605·9839	1·	1·	0·00165	0·00165	**7·1**
7·2	669·7150	669·7158	1·	1·	0·00149	0·00149	**7·2**
7·3	740·1496	740·1503	1·	1·	0·00135	0·00135	**7·3**
7·4	817·9919	817·9925	1·	1·	0·00122	0·00122	**7·4**
7·5	904·0209	904·0215	1·	1·	0·00111	0·00111	**7·5**

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